Well-Posedness of Parabolic Differential and Difference Equations with the Fractional Differential Operator

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Abstract

The stable difference scheme for the approximate solution of the initial value problem

$$\frac{du(t)}{dt}+D_t^{\frac{1}{2}}u(t)+Au(t)=f(t),\ 0<t<1,u(0)=0$$

for the differential equation in a Banach space E with the strongly positive operator A and fractional operator $D_t^{\frac{1}{2}}$ is presented. The well-posedness of the difference scheme in difference analogues of spaces of smooth functions is established. In practice, the coercive stability estimates for the solution of difference schemes for the 2m-th order multi-dimensional fractional parabolic equation and the one-dimensional fractional parabolic equation with nonlocal boundary conditions in space variable are obtained.

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1. INTRODUCTION. THE DIFFERENTIAL PROBLEM

It is known that differential equations involving derivatives of noninteger order have shown to be adequate models for various physical phenomena in areas like rheology, damping laws, diffusion processes, etc. (see, e.g., [1]- [11] and the references given therein). A review of some applications of fractional calculus in continuum and statistical mechanics is given by Mainardi [7].

The role played by coercive stability inequalities (well-posedness) in the study of boundary-value problems for parabolic partial differential and difference equations is well known (see, e.g.,[12], [15]). In paper [17], the initial value problem

$$\frac{du(t)}{dt}+D_t^{\frac{1}{2}}u(t)+Au(t)=f(t),\ 0<t<1,u(0)=0 \tag{1}$$

for the fractional differential equation in a Banach space E with the strongly positive operator A is considered. Here $D_t^{\frac{1}{2}}=D_{0+}^{\frac{1}{2}}$ is the standard Riemann-Lioville's derivative of order $\frac{1}{2}$. This fractional differential equation corresponds to the Basset problem [6]. It represents a classical problem in fluid dynamics where the unsteady motion of a particle accelerates in a viscous fluid due to the gravity of force.

A function $u(t)$ is called a solution of the problem (1) if the following conditions are satisfied:

i) $u(t)$ is continuously differentiable on the segment $[0,1]$,

ii) The element $u(t)$ belongs to $D(A)$ for all $t\in[0,1]$ and the function $Au(t)$ is continuous on the segment $[0,1]$,

iii) $u(t)$ satisfies the equation and the initial condition (1).

A solution of problem (1) defined in this manner will from now on referred to as a solution of problem (1) in the space $C(E)=C([0,1],E)$ of all continuous functions $\varphi(t)$ defined on $[0,1]$ with values in E equipped with the norm

$$||\varphi||_{C(E)}=\max_{0\le t\le 1}||\varphi(t)||_E.$$

The well-posedness in $C(E)$ of the boundary value problem (1) means that coercive inequality

$$\|u'\|_{C(E)} + \|D_t^{\frac{1}{2}}u\|_{C(E)} + \|Au\|_{C(E)} \leq M\|f\|_{C(E)}$$

is true for its solution $u(t) \in C(E)$ with some M, which does not depend on $f(t) \in C(E)$.

Positive constants, which can differ in time (hence: not a subject of precision) will be indicated with an M . On the other hand $M(\alpha, \beta, \cdots)$ is used to focus on the fact that the constant depends only on $\alpha, \beta, \cdots$.

In paper [17], the following theorems on well-posedness of (1) in spaces of smooth functions was established.

Theorem 1. Let A be a strongly positive operator in a Banach space E and $f(t) \in C(E)$. Then, for the solution $u(t)$ in $C(E)$ of the initial value problem (1) the stability inequality holds:

$$\| D_t^{\frac{1}{2}} u \|_{C(E)} + \| u' + Au \|_{C(E)} \leq M \| f \|_{C(E)} . \tag{2}$$

Theorem 2. Let A be a strongly positive operator in a Banach space E and $f(t) \in C(E_\alpha)(0 < \alpha < 1)$. Then for the solution $u(t)$ in $C(E_\alpha)$ of the initial value problem (1) the coercive inequality is valid:

$$\| u' \|_{C(E_\alpha)} + \| Au \|_{C(E_\alpha)} \leq M\alpha^{-1}(1-\alpha)^{-1} \| f \|_{C(E_\alpha)} . \tag{3}$$

Here, the fractional space $E_\alpha = E_\alpha(E, A)(0 < \alpha < 1)$, consisting of all $v \in E$ for which the following norm is finite:

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^{1-\alpha} \| Aexp(-\lambda A)v \|_E$$

is additionally introduced.

In the present paper, the stable difference scheme for the approximate solution of initial value problem (1)

$$\begin{cases} \tau^{-1}(u_k - u_{k-1}) + Au_k + D_\tau^{\frac{1}{2}} u_k = f_k, \\ f_k = f(t_k), \ t_k = k\tau, \ 1 \leq k \leq N, \ N\tau = 1, \ u_0 = 0 \end{cases} \tag{4}$$

is presented. Here,

$$D_\tau^{\frac{1}{2}} u_k = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!} \frac{u_m - u_{m-1}}{\tau^{\frac{1}{2}}}, \Gamma(k-m+\frac{1}{2}) = \int_0^\infty t^{k-m-\frac{1}{2}} e^{-t} dt. \tag{5}$$

The paper is organized as follows. The well-posedness of (4) in difference analogues of spaces of smooth functions is established in Section 2. In Section 3 the coercive stability estimates for the solution of difference schemes for the $2m$-th order multi-dimensional fractional parabolic equation and the one-dimensional fractional parabolic equation with nonlocal boundary conditions in space variable are obtained.

2. THE WELL-POSEDNESS OF DIFFERENCE SCHEME

Let us first obtain the representation for the solution of problem (4). It is clear that the first order of accuracy difference scheme

$$\tau^{-1}(u_k - u_{k-1}) + Au_k = F_k, \quad 1 \leq k \leq N, \ N\tau = 1, u_0 = 0$$

has a solution and the following formula holds:

$$u_k = \sum_{s=1}^{k} R^{k-s+1} F_s \tau,\ 1 \le k \le N,$$

where $R = (I + \tau A)^{-1}$. Applying the formula $F_k = f_k - D_\tau^{\frac{1}{2}} u_k$, we get

$$u_k = -\sum_{s=1}^{k} R^{k-s+1} D_\tau^{\frac{1}{2}} u_s \tau + \sum_{s=1}^{k} R^{k-s+1} f_s \tau,\ 1 \le k \le N. \tag{6}$$

So, formula (6) gives the representation for the solution of problem (4).
Let $F_\tau(E)$ be the linear space of mesh functions $\varphi^\tau = \{\varphi_k\}_1^N$ with values in the Banach space E. Next on $F_\tau(E)$ we introduce the Banach space $C_\tau(E) = C([0,1]_\tau, E)$ with the norm

$$\| \varphi^\tau \|_{C_\tau(E)} = \max_{1 \le k \le N} \| \varphi_k \|_E.$$

Theorem 3. Let A be a strongly positive operator in a Banach space E . Then, for the solution $u^\tau = \{u_k\}_1^N$ in $C_\tau(E)$ of initial value problem (4) the stability inequality holds:

$$\left\| \left\{ D_\tau^{\frac{1}{2}} u_k \right\}_1^N \right\|_{C_\tau(E)} + \left\| \left\{ \tau^{-1}(u_k - u_{k-1}) + A u_k \right\}_1^N \right\|_{C_\tau(E)} \le M \| f^\tau \|_{C_\tau(E)} . \tag{7}$$

Proof. Using formula (6), we get

$$\tau^{-1}(u_k - u_{k-1}) = -D_\tau^{\frac{1}{2}} u_k + \sum_{s=1}^{k} A R^{k-s+1} D_\tau^{\frac{1}{2}} u_s \tau + f_k - \sum_{s=1}^{k} A R^{k-s+1} f_s \tau. \tag{8}$$

Applying formulas (8) and (5), we obtain

$$D_\tau^{\frac{1}{2}} u_k = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{k} \frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!} \tau^{\frac{1}{2}} \left[-D_\tau^{\frac{1}{2}} u_m + f_m \right]$$

$$+ \frac{1}{\sqrt{\pi}} \sum_{s=1}^{k} \sum_{m=s}^{k} \frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!} A R^{m-s+1} D_\tau^{\frac{1}{2}} u_s \tau^{\frac{3}{2}}$$

$$- \frac{1}{\sqrt{\pi}} \sum_{s=1}^{k} \sum_{m=s}^{k} \frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!} A R^{m-s+1} f_s \tau^{\frac{3}{2}}.$$

Let us first obtain the estimate

$$\left\| \frac{1}{\sqrt{\pi}} \sum_{m=s}^{k} \frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!} A R^{m-s+1} \tau^{\frac{1}{2}} \right\|_{E \to E} \le \frac{M}{\sqrt{(k-s)\,\tau}} \tag{9}$$

for any $1 \le s < k \le N$. We have that

$$\frac{1}{\sqrt{\pi}} \sum_{m=s}^{k} \frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!} A R^{m-s+1} \tau^{\frac{1}{2}} = \frac{1}{\sqrt{\pi}} \sum_{m=\left[\frac{s+k}{2}\right]}^{k} \frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!} A R^{m-s+1} \tau^{\frac{1}{2}}$$

$$+\frac{1}{\sqrt{\pi}}\sum_{m=s}^{\left[\frac{s+k}{2}\right]-1}\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}AR^{m-s+1}\tau^{\frac{1}{2}}=J_1+J_2.$$

Using estimates [14]

$$\left\|AR^k\right\|_{E\to E}\le\frac{M}{k\tau},\left\|R^k\right\|_{E\to E}\le M,1\le k\le N \tag{10}$$

and the following elementary inequality

$$\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}\le\frac{1}{\sqrt{k-m}},0\le m<k, \tag{11}$$

we get

$$\|J_1\|_{E\to E}\le\frac{1}{\sqrt{\pi}}\sum_{m=\left[\frac{s+k}{2}\right]}^{k}\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}\left\|AR^{m-s+1}\right\|_{E\to E}\tau^{\frac{1}{2}} \tag{12}$$

$$\le\frac{2M}{(k-s)\,\tau}\frac{1}{\sqrt{\pi}}\sum_{m=\left[\frac{s+k}{2}\right]}^{k}\frac{\tau}{\sqrt{(k-m)\,\tau}}\le\frac{M_1}{\sqrt{(k-s)\,\tau}}.$$

Now, we shall estimate J_2. We have that

$$J_2=\frac{1}{\sqrt{\pi}}\frac{\Gamma(k-s+\frac{1}{2})}{(k-s)!}\tau^{-\frac{1}{2}}-\frac{1}{\sqrt{\pi}}\frac{\Gamma(k-\left[\frac{s+k}{2}\right]+\frac{3}{2})}{(k-\left[\frac{s+k}{2}\right]+1)!}R^{\left[\frac{s+k}{2}\right]-s}\tau^{-\frac{1}{2}}$$

$$+\frac{1}{\sqrt{\pi}}\sum_{m=s+1}^{\left[\frac{s+k}{2}\right]-1}\left[\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}-\frac{\Gamma(k-m+\frac{3}{2})}{(k-m+1)!}\right]R^{m-s}\tau^{-\frac{1}{2}}.$$

Applying estimates (10) and (11), we obtain

$$\|J_2\|_{E\to E}\le\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{(k-s)\,\tau}}+\frac{1}{\sqrt{\pi}}\left\|R^{\left[\frac{s+k}{2}\right]-s}\right\|_{E\to E}\frac{1}{\sqrt{(k-\left[\frac{s+k}{2}\right]+1)\tau}} \tag{13}$$

$$+\frac{1}{\sqrt{\pi}}\sum_{m=s+1}^{\left[\frac{s+k}{2}\right]-1}\left|\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}-\frac{\Gamma(k-m+\frac{3}{2})}{(k-m+1)!}\right|\left\|R^{m-s}\right\|_{E\to E}\tau^{-\frac{1}{2}}$$

$$\le\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{(k-s)\,\tau}}+\frac{1}{\sqrt{\pi}}M\frac{\sqrt{2}}{\sqrt{(k-s)\tau}}$$

$$+M\frac{\frac{1}{2}}{\sqrt{\pi}}\sum_{m=s+1}^{\left[\frac{s+k}{2}\right]-1}\frac{\tau}{(k-m+1)\,\tau\sqrt{(k-m)\,\tau}}\le\frac{M_2}{\sqrt{(k-s)\,\tau}}.$$

Estimate (9) follows from estimates (12) and (13).

Now, let us first estimate $z_k=\left\|D_\tau^{\frac{1}{2}}u_k\right\|_E$. Applying the triangle inequality and estimate (9), we get

$$z_k\le\frac{1}{\sqrt{\pi}}\sum_{m=1}^{k}\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}\tau^{\frac{1}{2}}\left[z_m+\|f_m\|_E\right]$$

$$+\frac{1}{\sqrt{\pi}}\sum_{s=1}^{k}\left\|\sum_{m=s}^{k}\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}AR^{m-s+1}\right\|_{E\to E} z_s\tau^{\frac{3}{2}}$$

$$+\frac{1}{\sqrt{\pi}}\sum_{s=1}^{k}\left\|\sum_{m=s}^{k}\frac{\Gamma(k-m+\frac{1}{2})}{(k-m)!}AR^{m-s+1}\right\|_{E\to E} \|f_s\|_E\,\tau^{\frac{3}{2}}$$

$$\leq M_3\sum_{s=1}^{k-1}\frac{1}{\sqrt{(k-s)\,\tau}}\tau\left[z_s+\|f_s\|_E\right]+M_4\left[z_k+\|f_k\|_E\right]\tau^{\frac{1}{2}}$$

for any $k=1,\cdot\cdot\cdot,N$. Applying the above inequality and the difference analogue of the integral inequality, we obtain

$$\left\|\left\{D_\tau^{\frac{1}{2}}u_k\right\}_1^N\right\|_{C_\tau(E)}\leq M\,\|f^\tau\|_{C_\tau(E)}\,. \tag{14}$$

Using the triangle inequality and equation (4), we get

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})+Au_k\right\}_1^N\right\|_{C_\tau(E)}\leq\left[\|f^\tau\|_{C_\tau(E)}+\left\|\left\{D_\tau^{\frac{1}{2}}u_k\right\}_1^N\right\|_{C_\tau(E)}\right]$$

$$\leq M_1\parallel f^\tau\parallel_{C_\tau(E)}\,. \tag{15}$$

Estimate (7) follows from estimates (14) and (15). Theorem 3 is proved.

Theorem 4. Let A be a strongly positive operator in a Banach space E. Then, for the solution $u^\tau=\{u_k\}_1^N$ in $C_\tau(E)$ of initial value problem (4) the almost coercive stability inequality is valid:

$$\left\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\right\|_{C_\tau(E)}+\left\|\{Au_k\}_1^N\right\|_{C_\tau(E)} \tag{16}$$

$$\leq M\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{E\to E}\right\}\parallel f^\tau\parallel_{C_\tau(E)}\,.$$

Proof. The proof of estimate

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\right\|_{C_\tau(E)}\leq M\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{E\to E}\right\}\parallel f^\tau\parallel_{C_\tau(E)} \tag{17}$$

for the solution of initial value problem (4) is based on estimate (7) and the following estimates [12]:

$$\max_{1\leq k\leq N}\left\|\sum_{s=1}^{k}AR^{k-s+1}f_s\tau\right\|_E\leq M\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{E\to E}\right\}\parallel f^\tau\parallel_{C(E)},$$

$$\max_{1\leq k\leq N}\left\|\sum_{s=1}^{k}AR^{k-s+1}D_\tau^{\frac{1}{2}}u_s\tau\right\|_E\leq M\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{E\to E}\right\}\parallel\{D_\tau^{\frac{1}{2}}u_k\}_1^N\parallel_{C(E)}\,.$$

Using these estimates, the triangle inequality and equation (4), we get

$$\left\|\{Au_k\}_1^N\right\|_{C_\tau(E)}\leq M_1\min\left\{\ln\frac{1}{\tau},1+\ln\|A\|_{E\to E}\right\}\parallel f^\tau\parallel_{C(E)}\,. \tag{18}$$

Estimate (16) follows from estimates (17) and (18). Theorem 4 is proved.

Note that the Banach space $E'_\alpha = E'_\alpha(E,A)(0<\alpha<1)$ consists of those $v\in E$ for which the norm

$$\| v \|_{E'_\alpha} = \sup_{\lambda>0} \lambda^\alpha \parallel A(\lambda+A)^{-1}v \parallel_E$$

is finite.

Theorem 5. Let A be a strongly positive operator in a Banach space E . Then, for the solution $u^\tau = \{u_k\}_1^N$ in $C_\tau(E'_\alpha)$ of the initial value problem (4) the coercive stability inequality is valid:

$$\left\|\{\tau^{-1}(u_k-u_{k-1})\}_1^N\right\|_{C_\tau(E'_\alpha)} + \left\|\{Au_k\}_1^N\right\|_{C_\tau(E'_\alpha)} \tag{19}$$
$$\le M\alpha^{-1}(1-\alpha)^{-1} \parallel f^\tau \parallel_{C_\tau(E'_\alpha)} .$$

Proof. By Theorem 3,

$$\left\|\left\{D_\tau^{\frac{1}{2}}u_k\right\}_1^N\right\|_{C_\tau(E'_\alpha)} \le M \parallel f^\tau \parallel_{C_\tau(E'_\alpha)} \tag{20}$$

for the solution of initial value problem (4). The proof of estimate

$$\left\|\left\{\tau^{-1}(u_k-u_{k-1})\right\}_1^N\right\|_{C_\tau(E'_\alpha)} \le M\alpha^{-1}(1-\alpha)^{-1} \parallel f^\tau \parallel_{C_\tau(E'_\alpha)} \tag{21}$$

for the solution of initial value problem (4) is based on estimate (20) and the following estimates [12] :

$$\max_{1\le k\le N}\left\|\sum_{s=1}^k AR^{k-s+1}f_s\tau\right\|_{E'_\alpha} \le M\alpha^{-1}(1-\alpha)^{-1} \parallel f^\tau \parallel_{C(E'_\alpha)}, \tag{22}$$

$$\max_{1\le k\le N}\left\|\sum_{s=1}^k AR^{k-s+1}D_\tau^{\frac{1}{2}}u_s\tau\right\|_{E'_\alpha} \le M\alpha^{-1}(1-\alpha)^{-1} \parallel \{D_\tau^{\frac{1}{2}}u_k\}_1^N \parallel_{C(E'_\alpha)} . \tag{23}$$

Using the triangle inequality, estimates (22), (23) and equation (4), we get

$$\left\|\{Au_k\}_1^N\right\|_{C_\tau(E'_\alpha)} \le M_1\alpha^{-1}(1-\alpha)^{-1} \parallel f^\tau \parallel_{C(E'_\alpha)} . \tag{24}$$

Estimate (19) follows from estimates (21) and (24). Theorem 5 is proved.

Note that by passing to the limit for $\tau\to0$ one can recover Theorems 1 and 2.

3. APPLICATIONS

Now, we consider the applications of Theorem 3, 4 and 5.

First, the initial-value problem on the range $\{0\le t\le 1, x\in \mathrm{R}^n\}$ for the $2m$-order multi-dimensional fractional parabolic equation is considered:

$$\begin{cases} \frac{\partial v(t,x)}{\partial t} + D_t^{\frac{1}{2}}v(t,x) + \sum\limits_{|r|=2m} a_r(x)\frac{\partial^{|r|}v(t,x)}{\partial x_1^{r_1}\cdots\partial x_n^{r_n}} + \sigma v(t,x) = f(t,x), \\ 0<t<1; v(0,x)=0, x\in\mathrm{R}^n, |r| = r_1+\cdots+r_n, \end{cases} \tag{25}$$

where $a_r(x)$ and $f(t,x)$ are given as sufficiently smooth functions. Here, σ is a sufficiently large positive constant.

The discretization of problem (25) is carried out in two steps. In the first step, the grid space $\mathrm{R}_h^n\,(0<h\leq h_0)$ is defined as the set of all points of the Euclidean space R^n whose coordinates are given by

$$x_k = s_k h, \qquad s_k = 0, \pm 1, \pm 2, \cdots, k = 1, \cdots, n.$$

The difference operator $A_h^x = B_h^x + \sigma I_h$ is assigned to the differential operator $A^x = B^x + \sigma I$, defined by (25). The operator

$$B_h^x = h^{-2m} \sum_{2m\leq|s|\leq S} b_s^x \Delta_{1-}^{s_1} \Delta_{1+}^{s_2} ... \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}}, \tag{26}$$

acts on functions defined on the entire space R_h^n. Here, $s \in \mathrm{R}^{2\mathrm{n}}$ is a vector with nonnegative integer coordinates,

$$\Delta_{k\pm} f^h(x) = \pm \left(f^h(x \pm e_k h) - f^h(x) \right),$$

where e_k is the unit vector of the axis x_k.
An infinitely differentiable function $\varphi(x)$ of the continuous argument $x \in \mathrm{R}^n$ that is continuous and bounded together with all its derivatives is said to be smooth. We say that the difference operator A_h^x is a λ-th order $(\lambda > 0)$ approximation of the differential operator A^x if the inequality

$$\sup_{x\in R_h^n} |A_h^x \varphi(x) - A^x \varphi(x)| \leq M(\varphi) h^\lambda$$

holds for any smooth function $\varphi(x)$. The coefficients b_s^x are chosen in such a way that the operator A_h^x approximates in a specified way the operator A^x. It is assumed that the operator A_h^x approximates the differential operator A^x with any prescribed order [16] .

The function $A^x(\xi h, h)$ is obtained by replacing the operator $\Delta_{k\pm}$ in the right-hand side of equality (26) with the expression $\pm(\exp\{\pm i\xi_k h\} - 1)$, respectively, and is called the symbol of the difference operator B_h^x.
We shall assume that for $|\xi_k h| \leq \pi$ and fixed x the symbol $A^x(\xi h, h)$ of the operator $B_h^x = A_h^x - \sigma I_h$ satisfies the inequalities

$$(-1)^m A^x(\xi h, h) \geq M|\xi|^{2m}, |\arg A^x(\xi h, h)| \leq \phi < \phi_0 \leq \frac{\pi}{2}. \tag{27}$$

Suppose that the coefficient b_s^x of the operator $B_h^x = A_h^x - \sigma I_h$ is bounded and satisfies the inequalities

$$|b_s^{x+e_k h} - b_s^x| \leq M h^\epsilon, x \in \mathrm{R}_h^n, \epsilon \in (0,1]. \tag{28}$$

With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{dv^h(t,x)}{dt} + D_t^{\frac{1}{2}} v^h(t,x) + A_h^x v^h(t,x) = f^h(t,x), \ 0 < t < 1, \ x \in \mathbb{R}_h^n, \\ v^h(0,x) = 0, x \in \mathbb{R}_h^n \end{cases} \tag{29}$$

for an infinite system of ordinary differential equations.
In the second step, problem (29) is replaced by the difference scheme

$$\begin{cases} \frac{u_k^h(x) - u_{k-1}^h(x)}{\tau} + D_\tau^{\frac{1}{2}} u_k^h + A_h^x u_k^h = f_k^h(x), \\ f_k^h(x) = f^h(t_k, x), t_k = k\tau, 1 \leq k \leq N, \ N\tau = 1, \ x \in \mathbb{R}_h^n, \\ u_0^h(x) = 0, \ x \in \mathbb{R}_h^n. \end{cases} \tag{30}$$

Based on the number of corollaries of the abstract theorems given in the above, to formulate the result, one needs to introduce the spaces $C_h = C(R_h^n)$ and $C_h^\beta = C^\beta(\mathrm{R}_h^n)$ of all bounded grid functions $u^h(x)$ defined on R_h^n, equipped with the norms

$$||u^h||_{C_h} = \sup_{x\varepsilon\mathrm{R}_h^n} |u^h(x)|, ||u^h||_{C_h^\beta} = \sup_{x\varepsilon\mathrm{R}_h^n} |u^h(x)| + \sup_{x,y\in\mathrm{R}_h^n} \frac{|u^h(x) - u^h(x+y)|}{|y|^\beta}.$$

Theorem 6. Suppose that assumptions (27) and (28) for the operator A_h^x hold. Then, the solutions of the difference scheme (30) satisfy the following stability estimates:

$$\max_{1\le k\le N} \left\| D_\tau^{\frac{1}{2}} u_k^h \right\|_{C_h^\mu} \le M_1(\mu) \max_{1\le k\le N} \left\| f_k^h \right\|_{C_h^\mu}, 0 \le \mu \le 1,$$

$$\left\| \left\{ \tau^{-1}(u_k^h - u_{k-1}^h) \right\}_1^N \right\|_{C_\tau(C_h^\mu)} \le M(\mu) \ln \frac{1}{\tau + |h|} \max_{1\le k\le N} \left\| f_k^h \right\|_{C_h^\mu}, 0 \le \mu \le 1,$$

$$\left\| \left\{ \tau^{-1}(u_k^h - u_{k-1}^h) \right\}_1^N \right\|_{C_\tau(C_h^{\mu+2m\alpha})} \le M(\alpha,\mu) \max_{1\le k\le N} \left\| f_k^h \right\|_{C_h^{\mu+2m\alpha}}, 0 < 2m\alpha + \mu < 1.$$

The proof of Theorem 5 is based on the abstract Theorems 3, 4 and 5, the strongly positivity of the operator A_h^x defined by (33) in C_h^μ [16] and the estimate

$$\min \left\{ \ln \frac{1}{\tau}, 1 + \ln \|A_h^x\|_{C_h^\mu \to C_h^\mu} \right\} \le M(\mu) \ln \frac{1}{\tau + |h|}$$

and on the following two theorems on the coercivity inequality for the solution of the elliptic difference equation in C_h^β and on the structure of the fractional space $E'_\alpha(C_h, A_h^x)$.

Theorem 7[12]. Suppose that assumptions (27) and (28) for the operator A_h^x hold. Then, for the solution of the elliptic difference equation

$$A_h^x u^h(x) = \omega^h(x), x \in \mathrm{R}_h^n \tag{31}$$

the estimate

$$\sum_{2m\le|s|\le S} h^{-2m} ||\Delta_{1-}^{s_1} \Delta_{1+}^{s_2} ... \Delta_{n-}^{s_{2n-1}} \Delta_{n+}^{s_{2n}} u^h||_{C_h^\beta} \le M(\sigma,\beta) ||\omega^h||_{C_h^\beta}$$

is valid.

Theorem 8[12]. Suppose that assumptions (27) and (28) for the operator A_h^x hold. Then, for any $0 < \alpha < \frac{1}{2m}$ the norms in the spaces $E'_\alpha(C_h, A_h^x)$ and $C_h^{2m\alpha}$ are equivalent uniformly in h.

Second, we consider the mixed boundary value problem for the fractional parabolic equation

$$\left\{ \begin{array}{c} \frac{\partial v(t,x)}{\partial t} + D_t^{\frac{1}{2}} v(t,x) - a(x) \frac{\partial^2 v(t,x)}{\partial x^2} + \sigma v(t,x) = f(t,x), \\ 0 < t < 1, 0 < x < 1; v(0,x) = 0, 0 \le x \le 1, \\ u(t,0) = u(t,1), \quad u_x(t,0) = u_x(t,1), \quad 0 \le t \le 1, \end{array} \right. \tag{32}$$

where $a(t,x)$ and $f(t,x)$ are given sufficiently smooth functions and $a(t,x) \ge a > 0$. Here, σ is a sufficiently large positive constant.

The discretization of problem (32) is carried out in two steps. In the first step, let us define the grid space

$$[0,1]_h = \{x : x_r = rh, 0 \le r \le K, Kh = 1\}.$$

We introduce the Banach space $C_h^\beta = C^\beta([0,1]_h)$ $(0<\beta<1)$ of the grid functions $\varphi^h(x) = \{\varphi_r\}_1^{K-1}$ defined on $[0,1]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{C_h^\beta} = \left\|\varphi^h\right\|_{C_h} + \sup_{1\le k<k+\tau\le K-1} \frac{|\varphi_{k+r}-\varphi_k|}{\tau^\beta},$$

where $C_h = C([0,1]_h)$ is the space of the grid functions $\varphi^h(x) = \{\varphi_r\}_1^{K-1}$ defined on $[0,1]_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{C_h} = \max_{1\le k\le K-1} |\varphi_k|.$$

To the differential operator A generated by the problem (32), we assign the difference operator A_h^x by the formula

$$A_h^x\varphi^h(x) = \left\{-(a(x)\varphi_{\bar{x}})_{x,r} + \delta\varphi_r\right\}_1^{K-1}, \tag{33}$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_r\}_0^K$ satisfying the conditions $\varphi_0 = \varphi_K$, $\varphi_1 - \varphi_0 = \varphi_K - \varphi_{K-1}$. With the help of A_h^x, we arrive at the initial value problem

$$\begin{cases} \frac{dv^h(t,x)}{dt} + D_t^{\frac{1}{2}} v^h(t,x) + A_h^x v^h(t,x) = f^h(t,x),\ 0<t<1,\ x\in[0,1]_h, \\ v^h(0,x) = 0, x\in[0,1]_h \end{cases} \tag{34}$$

for an infinite system of ordinary fractional differential equations. In the second step, we replace problem (34) by difference scheme (4)

$$\begin{cases} \frac{u_k^h(x)-u_{k-1}^h(x)}{\tau} + D_\tau^{\frac{1}{2}} u_k^h + A_h^x u_k^h(x) = f_k^h(x),\ f_k^h(x) = \{f(t_k,x_r)\}_1^{K-1}, \\ t_k = k\tau, 1\le k\le N,\ N\tau=1; u_0^h(x)=0,\ x\in[0,1]_h. \end{cases} \tag{35}$$

Theorem 9. Let τ and h be sufficiently small numbers. Then, the solutions of the difference scheme (35) satisfy the following stability estimates:

$$\max_{1\le k\le N}\left\|D_\tau^{\frac{1}{2}} u_k^h\right\|_{C_h^\mu} \le M_1(\mu) \max_{1\le k\le N}\left\|f_k^h\right\|_{C_h^\mu}, 0\le\mu\le 1,$$

$$\left\|\left\{\tau^{-1}(u_k^h - u_{k-1}^h)\right\}_1^N\right\|_{C_\tau(C_h^\mu)} \le M(\mu)\ln\frac{1}{\tau+h}\max_{1\le k\le N}\left\|f_k^h\right\|_{C_h^\mu}, 0\le\mu\le 1,$$

$$\left\|\{\tau^{-1}(u_k^h - u_{k-1}^h)\}_1^N\right\|_{C_\tau(C_h^{\mu+2\alpha})} \le M(\alpha,\mu)\max_{1\le k\le N}\left\|f_k^h\right\|_{C_h^{\mu+2\alpha}}, 0<2\alpha+\mu<1.$$

The proof of Theorem 9 is based on the abstract Theorems 3, 4 and 5, the strongly positivity of the operator A_h^x defined by (33) in C_h^μ and the estimate

$$\min\left\{\ln\frac{1}{\tau}, 1+\ln\|A_h^x\|_{C_h^\mu\to C_h^\mu}\right\} \le M(\mu)\ln\frac{1}{\tau+h}$$

and on the following theorem on the structure of the fractional space $E'_\alpha(C_h, A_h^x)$.
Theorem 10 [13]. For any $0<\alpha<\frac{1}{2}$, the norms in the spaces $E'_\alpha(C_h, A_h^x)$ and $C_h^{2\alpha}$ are equivalent uniformly in h.

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