On non commutative Taylor invertibility

Robin Harte

Trinity College, Dublin; rharte@maths.tcd.ie

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Abstract *For general non commutative systems of Banach algebra elements, the one way "forward" spectral mapping theorem fails for the Taylor split spectrum.*

0. Introduction

The extension of spectral theory from single elements to finite or infinite systems is mostly confined to *commuting* systems, although usually the definitions survive without this restriction. In a linear algebra, or more generally a "linear category" A, a *spectrum* $\omega(a) \subseteq \mathbf{C}^X$ is derived from some collection $H \subseteq A$ of "invertible" or more generally non-singular, systems of elements $a \in A^X$:

0.1 $$\omega(a) = \{\lambda \in \mathbf{C}^X : a - \lambda \notin H\} \ .$$

For such a "joint spectrum" we look for the *spectral mapping theorem*

0.2 $$p\omega(a) = \omega p(a) \subseteq \mathbf{C}^Y \ ,$$

for $a \in A^X$ and systems $p \in \mathrm{Poly}_X^Y$ of "non commutative polynomials". Equality (0.2) divides into a *forward* spectral mapping theorem,

0.3 $$p\omega(a) \subseteq \omega p(a) \ ,$$

and a *backward* spectral mapping theorem,

0.4 $$\omega p(a) \subseteq p\omega(a) \ .$$

Typically the forward theorem (0.3) is easier, and survives for other than commutative systems of elements, combining the remainder theorem for non commutative polynomials with some kind of reverse semi-group property of the non singulars H; the harder backward theorem (0.4) needs the "fundamental theorem of algebra", or more generally Liouville's theorem from complex analysis. In the present note however we observe that, for general non commuting systems, the forward spectral mapping theorem (0.3) is liable to fail for the Taylor spectrum.

1. Taylor invertibility

Suppose $a \in A$ and $b \in A$, for a complex linear algebra A with identity 1; then we shall say that the pair $(a,b) \in A^2$ is *Taylor invertible* if it is at once *left*, *right* and *middle invertible*: here

1.1 $$(a,b) \in A^{-2}_{left} \Longleftrightarrow 1 \in (\,A \quad A\,)\begin{pmatrix} a \\ b \end{pmatrix} \ ;$$

1.2 $$(a,b) \in A^{-2}_{right} \Longleftrightarrow 1 \in (\,b \quad -a\,)\begin{pmatrix} A \\ A \end{pmatrix} \ ;$$

1.3 $$(a,b) \in A^{-2}_{middle} \Longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} a \\ b \end{pmatrix}(\,A \quad A\,) + \begin{pmatrix} A \\ A \end{pmatrix}(\,b \quad -a\,) \ .$$

Necessary for left invertibility is the implication, for arbitrary $x \in A$,

1.4 $$ax = bx = 0 \Longrightarrow x = 0 \ ;$$

and for right invertibility is the implication, for arbitrary $y \in A$,

1.5 $$ya = yb = 0 \Longrightarrow y = 0 \ .$$

If the elements a, b commute, in the sense $ab = ba$, and A is a Banach algebra, then (1.1)-(1.3) add up to the condition that $(0,0)$ is not in the "Taylor split spectrum" of the pair (a,b): the point here is that we are witholding commutivity. Without commutivity there is a "one way spectral mapping theorem" for left and for right invertibility, and it would be nice to be able to say the same for "Taylor invertibility". The sequence of matrices

1.6 $$\left(0, (\,b \quad -a\,), \begin{pmatrix} a \\ b \end{pmatrix}, 0\right)$$

may [17],[12],[5],[8] be referred to as the *Koszul complex* of the pair $(a,b) \in A^2$; of course it will not truly be a "complex" unless

1.7 $$(\,b \quad -a\,)\begin{pmatrix} a \\ b \end{pmatrix} \equiv ba - ab = 0 \ ,$$

which says that a and b commute.

2. Matrices

If $A = \mathbf{C}^{2\times 2}$ and

2.1
$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ ,$$

then for arbitrary $(\lambda, \mu) \in \mathbf{C}^2$ the conditions (1.1) and (1.2) are satisfied by $(a, b) = (e - \lambda, f - \mu)$: in words both the left and the right spectrum of the pair (e, f) are [5],[8] empty. It is also true that the condition (1.3) is satisfied unless $(\lambda, \mu) = (0, 0)$: however

2.2
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \notin \begin{pmatrix} e \\ f \end{pmatrix} (\, A \quad A\,) + \begin{pmatrix} A \\ A \end{pmatrix} (\, f \quad -e\,) = \begin{pmatrix} eA + Af & eA + Ae \\ fA + Af & fA + Ae \end{pmatrix} \ ,$$

and hence the right hand side of (1.3) is equivalent to inclusion

2.3
$$1 \in (eA + Af) \cap (Ae + fA) \ ,$$

which says that each of the pairs (f, e) and (e, f) are "splitting exact". However each of the pairs (f, e) and (e, f) are [9] "skew exact", so that if they were also exact then e and f would have to be left or right invertible. Alternatively notice

2.4
$$(\, f \quad -e\,) \begin{pmatrix} f \\ e \end{pmatrix} = 0 \ ; \ \begin{pmatrix} f \\ e \end{pmatrix} \notin \begin{pmatrix} e \\ f \end{pmatrix} A \ .$$

Indeed

2.5
$$(\, f \quad -e\,) \begin{pmatrix} f \\ e \end{pmatrix} = f^2 - e^2 = 0 - 0 \ ,$$

while

2.6
$$\begin{pmatrix} f \\ e \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix} g \Longrightarrow g = (fe + ef)g = f^2 + e^2 = 0 \ ; \Longrightarrow f = e = 0 \ .$$

Thus the middle spectrum, and hence the Taylor spectrum, of this pair does contain a point, and is given by the singleton $\{(0, 0)\}$. From one point of view this might seem to be a good thing: the Taylor spectrum of this unruly pair of matrices is nonempty. There are however consequences: without commutivity, the "one way" spectral mapping theorem (0.3) now fails for the Taylor spectrum.

3. Spectral mapping theorems

Suppose $p \equiv p(z_1, z_2) \in \text{Poly}_2$ is a "polynomial" in two free variables, with in particular $p(0,0) = 0$: then in general, with no assumption of commutivity, there is implication

3.1 $$1 \in Ap(a,b) \Longrightarrow 1 \in \begin{pmatrix} A & A \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

and

3.2 $$1 \in p(a,b)A \Longrightarrow 1 \in \begin{pmatrix} b & -a \end{pmatrix} \begin{pmatrix} A \\ A \end{pmatrix} \ ;$$

thus if in particular $p(a,b) \in A^{-1}$ is invertible then $(a,b) \in A^2$ is both left and right invertible. In general however this may not be enough to ensure middle invertibility. Indeed

3.3 $$p = z_2 z_1 + z_1 z_2 \ , \ (a,b) = (e,f) \ \Longrightarrow \ p(a,b) = 1 \notin (aA + Ab) \cup (Aa + bA) \ ,$$

which implies that the right hand side of (1.3) cannot hold. With

3.4 $$\sigma^{left}(a,b) = \{(\lambda, \mu) \in \mathbf{C}^2 : (a - \lambda, b - \mu) \notin A^{-2}_{left}\} \ ,$$

3.5 $$\sigma^{right}(a,b) = \{(\lambda, \mu) \in \mathbf{C}^2 : (a - \lambda, b - \mu) \notin A^{-2}_{right}\}$$

and

3.6 $$\sigma^{middle}(a,b) = \{(\lambda, \mu) \in \mathbf{C}^2 : (a - \lambda, b - \mu) \notin A^{-2}_{middle}\} \ ,$$

(3.1) and (3.2) give inclusions

3.7 $$p\sigma^{left}(a,b) \subseteq \sigma^{left} p(a,b)$$

and

3.8 $$p\sigma^{right}(a,b) \subseteq \sigma^{right} p(a,b) \ .$$

However, with $(a,b) = (e,f)$ and $p = z_2 z_1 + z_1 z_2$ we have

3.9 $$p\sigma^{middle}(a,b) = \{p(0,0)\} = \{0\} \not\subseteq \{1\} = \sigma p(a,b) \ .$$

4. Shifts

For further examples of such misbehaviour we might recall the backward and forward *shifts*. If for example $ba \in A^{-1}$, $(a, b) \in A^2$ is both left and right invertible: with

4.1 $$cba = 1 = bac \ ,$$

it is clear that

4.2 $$\begin{pmatrix} cb & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 1 = \begin{pmatrix} b & -a \end{pmatrix} \begin{pmatrix} ac \\ 0 \end{pmatrix} \ .$$

If ba and ab are both invertible then $(a, b) \in A^2$ will also be middle invertible; generally however there is equality

4.3 $$\begin{pmatrix} b'' \\ a'' \end{pmatrix} \begin{pmatrix} b & -a \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} a' & b' \end{pmatrix} \equiv \begin{pmatrix} aa' + b''b & ab' - b''a \\ ba' + a''b & bb' - a''a \end{pmatrix} \ ,$$

and then equivalence

4.4 $$\begin{pmatrix} aa' + b''b & ab' - b''a \\ ba' + a''b & bb' - a''a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longleftrightarrow \begin{matrix} aa' + b''b = 1 \ ; & ab' - b''a = 0 \ ; \\ ba' + a''b = 0 \ ; & bb' - a''a = 1 \ . \end{matrix}$$

If for example

4.5 $$ba = 1 \neq ab$$

then the top left hand condition on the right and side of (4.3) can easily fail; the condition (4.1) holds with $c = 1$, while there are $(x, y) \in A^2$ violating an obvious necessary condition for (1.3):

4.6 $$x = y = 1 - ab \Longrightarrow bx = ya = 0 \neq yx \ ,$$

and hence $1 \notin aA + Ab$. Alternatively take $(a, b) = (e, f)$ and $(x, y) = (f, e) = (b, a)$.

5. Exactnesss

More general than either left or right invertibility is *self exactness*. We shall say that the pair $(b, a) \in A^2$ is *splitting exact*, and write

5.1 $$(b, a) \in A^{-(1,1)}_{left,right} \ ,$$

provided

5.2 $$1 \in Ab + aA \ .$$

More generally (*cf* [7]) we might write

5.3 $$A^{-(m,n)}_{left,right} = \{(b, a) \in A^m \times A^n : 1 \in A^m \cdot b + a \cdot A^n \equiv \sum_{k=1}^{m} Ab_k + \sum_{j=1}^{n} a_j A\} \ .$$

Now $a \in A$ is to be *self exact* provided (a, a) is exact:

5.4 $$A^{-1}_{left,right} = \{a \in A : (a, a) \in A^{-(1,1)}_{left,right}\} \ ,$$

and more generally

5.5 $$A^{-n}_{left,right} = \{a \in A^n : 1 \in A^n \cdot a + a \cdot A^n\} \ .$$

Exactness (5.1) makes sense in a ring; in a more general additive category it is necessary that

5.6 $$\exists \ ba \in A \ ,$$

the product is defined. We do not however include the requirement that the *chain condition*

5.7 $$ba = 0 \in A$$

is satisfied; for some readers therefore (5.1) might be referred to as "non commutative exactness". Self exactness in a linear algebra would seem to generate another kind of spectrum, writing, for $a \in A^n$,

5.8 $$\sigma^{left,right}(a) = \{\lambda \in \mathbf{C}^n : 1 \notin A^n \cdot (a - \lambda) + (a - \lambda) \cdot A^n\} \ .$$

We can now enquire whether (0.3) or (0.4) hold with $\omega = \sigma^{left,right}$. For the forward version (0.3) observe that if $p \in \mathrm{Poly}^m_n$ with $p(0) = 0$, there is inclusion

5.9 $$A^m \cdot p(a) + p(a) \cdot A^m \subseteq A^n \cdot a + a \cdot A^n \ .$$

Notice however that, with e and f as in (2.1),

5.10 $$\sigma^{left,right}(e) = \sigma^{left,right}(f) = \emptyset \ ;$$

$(Ae)^{-1}Ae = Ae + eA$ is the set of upper triangles, and $(Af)^{-1}Af = Af + fA$ the lower triangles.

Generally if $N \subseteq A$ is a subring then, in search for a "projection property", and hence (0.4), we borrow from some approximation theory [11],[14],[15] a sort of *residual quotient* [10],[11],[14], and define

5.11 $$N : N = \{c \in A : Nc + cN \subseteq N\} \ ;$$

now $N \subseteq N : N$ is a two-sided ideal and we can form the quotient $(N : N)/N$. Provided $1 \notin N$ then $N \subseteq A$ will be a proper two-sided ideal of the ring $N : N$; if further A is a Banach algebra and $N = \mathrm{cl}(N)$ is closed then $B = (N : N)/N$ is a non trivial Banach algebra in its own right. Now if $c \in \mathrm{comm}(N) \subseteq N : N$ then

5.12 $$\lambda \in \partial\sigma_B[c]_N \Longrightarrow 1 \notin N + A(c - \lambda) + (c - \lambda)A \ ;$$

if $\lambda_n \to \lambda$ with $[c - \lambda_n]_N \in B^{-1}$ then $\lambda \notin \sigma_B[c]_N$:

5.13 $$1 \in c'c + cc'' + N \Longrightarrow \|[c - \lambda_n]^{-1}_N\| \leq (\|c'\| + \|c''\|)\|[c]_N[c - \lambda_n]^{-1}_N\| \ .$$

6. Koszul matrices

The problem for the "left,right invertibility" of (5.8) is that it is not clear, for $a \in A^n$, that $N = A^n \cdot a + a \cdot A^n \subseteq A$ is a subring, closed under multiplication. In a Banach algebra A it is also not clear that it is norm closed; we would like, for $N = A^n \cdot a + a \cdot A^n$, implication

6.1 $$1 \in \mathrm{cl}(N) \Longrightarrow 1 \in N \ .$$

For "Taylor invertibility" the self exactness is applied not directly to the primary element $a \in A$ or system $a \in A^n$, but rather to its *Koszul matrix*. It is possible [12],[16] to pile up the Koszul complex of an n tuple $a \in A^n$ of linear algebra elements into a single matrix Λ_a in a larger algebra D, which is now potentially self exact; the definition is inductive. For a single element $a \in A$, whose Koszul complex is just the triple $(0, a, 0)$ we set

6.2 $$\Lambda_a = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} \ ;$$

we could alternatively make an "upper triangular" version. Inductively define, with $b \in A^k$ and $c \in A$,

6.3 $$\Lambda_{(b,c)} = \begin{pmatrix} \Lambda_b & O \\ \triangle_c & -\Lambda_b \end{pmatrix} \in D^{2\times 2} \ ,$$

where

6.4 $$\Lambda_b \in D \ , \ \triangle_c \in D$$

are respectively what has already been defined, and the block diagonal generated by the single element $c \in A$. Generally if $a \in A^n$ and $p \in \mathrm{Poly}_n^m$ we define

6.5 $$\Lambda_p \Lambda_a = \Lambda_{p(a)} \ .$$

Inductively we claim, for $a \in A^n$, that

6.6 $$a \textit{ commutative} \Longleftrightarrow \Lambda_a^2 = O \ :$$

note that if $c \in \mathrm{comm}(b) \subseteq A$ then $\triangle_c \in \mathrm{comm}(\Lambda_b) \subseteq D$ and

$$\Lambda_b^2 = O \in D \Longrightarrow \Lambda_{(b,c)}^2 = O \in D^{2\times 2} \ .$$

We now claim that

6.7 $$(a, a') \in A^{2n} \textit{ commutative} \ , \ a' \cdot a = 1 \in A \Longrightarrow I \in D\Lambda_a + \Lambda_a D \subseteq D \ ,$$

so that Λ_a is splitting self exact. We again argue by induction: if (6.7) holds with $a = b \in A^k$ then it continues to hold with $a = (b, c) \in A^k \times A$. In turn if $a \in A^n$ is commutative and $p \in \mathrm{Poly}_n$ with $p(0) = 0$ then there is $q \in \mathrm{Poly}_n^n$ with $p(a) = q(a) \cdot a$ and hence

6.8 $$p(a) \in A^{-1} \Longrightarrow I \in D\Lambda_a + \Lambda_a D \ ,$$

and the extension to $p \in \mathrm{Poly}_n^m$ is induction on m. Next

6.9 $$I \in D\Lambda_b + \Lambda_b D \subseteq D \Longleftrightarrow \begin{pmatrix} I & O \\ O & I \end{pmatrix} \in \begin{pmatrix} \Lambda_b & O \\ \triangle_c & -\Lambda_b \end{pmatrix} D^{2\times 2} + D^{2\times 2} \begin{pmatrix} \Lambda_b & O \\ \triangle_c & -\Lambda_b \end{pmatrix} \ ,$$

and we look for $\lambda \in \mathbf{C}$ for which

6.10 $$I \notin D\Lambda_b + \Lambda_b D \subseteq D \Longrightarrow \begin{pmatrix} I & O \\ O & I \end{pmatrix} \notin \begin{pmatrix} \Lambda_b & O \\ \triangle_{c-\lambda} & -\Lambda_b \end{pmatrix} D^{2\times 2} + D^{2\times 2} \begin{pmatrix} \Lambda_b & O \\ \triangle_{c-\lambda} & -\Lambda_b \end{pmatrix} \ .$$

Implication (6.1), and the multiplicative property $N \cdot N \subseteq N$ are clear when $N = Aa + aA \subseteq A$ is replaced by $N = D\Lambda_a + \Lambda_a D \subseteq D$ and extend from $\Lambda_b \in D$ to $\Lambda_{(b,c)} \in D^{2\times 2}$ whenever $c \in \mathrm{comm}(b) \subseteq A$.

7. Quasicommutivity

One of the reasons for at least trying to state problems for non commutative systems is the feeling that the commutative theory ought to extend to *quasicommutative* systems [4],[5],[8]: associated with $(a,b,c) \in A^3$ with

7.1 $$[a,b] \equiv ab - ba = c \ ; \ [a,c] = 0 = [b,c]$$

we have

7.2 $$0, \begin{pmatrix} a & -b & c \end{pmatrix}, \begin{pmatrix} b & c & 0 \\ a & 0 & c \\ -1 & -a & b \end{pmatrix}, \begin{pmatrix} c \\ -b \\ -a \end{pmatrix}, 0 \ .$$

If we can argue that, since we have a true complex here, the spectral mapping theorem holds for the Taylor (split) spectrum (in particular for one polynomial in three variables), then two things will follow: the spectral mapping theorem for the quasicommuting pair (a,b), and hence also quasinilpotency for the commutator $ab - ba$. A new challenge would be to relax the quasicommutativity to commutivity $ac = ca$ and still have a "spectral" proof that $\sigma(ab - ba) = \{0\}$. If we extend the definition of "quasicommutative" from n tuples $a \in A^n$ to arbitrary systems $a \in A^X$, in particular to A itself, then A is "quasicommutative" iff

7.3 $$[A,[A,A]] = \{0\} \ ;$$

explicitly, for arbitrary $(a,b,c) \in A^3$,

7.4 $$(ab - ba)c = c(ab - ba) \ .$$

Evidently Gelfand's theorem holds for quasicommutative Banach algebras. More generally, according to Feinstein, the spectral mapping theorem holds for Banach algebras which are *nilpotent* Lie [3]. The idea of Boasso seems to be generally to consider

7.5 $$\sigma(a, b, ab - ba) \ .$$

Instead of relaxing the commutivity of $a \in A^n$, Wawrzyńczyk [18],[19] has extended the projection property of the left spectrum to locally convex *Waelbroeck algebras* A, for which the invertible group $A^{-1} \subseteq A$ is topologically open and the inversion map z^{-1} continuous; it is tempting to try [20] and do the same thing for the Taylor split spectrum.

Another problem, in either Banach or Waelbroeck algebras A, would be to extend Allan's theorem [1]: if $G \subseteq \mathbf{C}$ is an open connected set, and if

7.6 $$\lambda \in G \Longrightarrow 1 \in A(a - \lambda) + (a - \lambda)A \subseteq A \ ,$$

does the holomorphic function $a - z : G \to A$ have a holomorphic left,right inverse $a^\wedge : G \to A$, for which

7.7 $$a^\wedge(z)(a - z) + (a - z)a^\wedge(z) \equiv 1 : G \to A \ ?$$

8. Determinant and adjugate

Comparing the Koszul complex of a commuting system with that of its polynomial image offers [6] a curious way to reach determinants and adjugates of operator matrices. For example if $a = (b, c) \in A^2$ is a pair of single elements then their Koszul complex (1.6) is given by $(0, T^\sim, T, 0)$ where

8.1 $$T = \begin{pmatrix} b \\ c \end{pmatrix} \ , \ T^\sim = \begin{pmatrix} c & -b \end{pmatrix} \ ;$$

if we then attack (b, c) with a pair (p, q) in Poly_2 of two-variable polynomials without constant term, then we will replace T and $T^\sim$ with S and $S^\sim$ which will be derived from T and $T^\sim$:

8.2 $$S = UT \ ; \ S^\sim U = |U|T \ ; \ S^\sim = T^\sim U^\sim \ ; \ U^\sim S = T|U| \ .$$

With

8.3 $$R^\sim S = 1 = S^\sim R \ ,$$

and

8.4 $$RS^\sim + SR^\sim = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

we get left and right invertibiity for (b, c);

8.5 $$(R^\sim U)T = 1 = T^\sim(U^\sim R) \ ;$$

and then also

8.6 $$RT^\sim U^\sim + UTR^\sim = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ .$$

If now, using (6.7) and (6.8) we could replace (8.6) by

8.7 $$R^\sim UT^\sim + TR^\sim U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

then $(T^\sim, T)$ would be exact and hence (b, c) would also be middle invertible.

If in (8.2) we have

8.8 $$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

then [6], fixing U as (8.8), with mutually commuting (u_{ij}) and varying b, c in $\mathrm{comm}\{u_{ij}\}$, (8.2) is uniquely satisfied by

8.9 $$|U| = u_{11}u_{22} - u_{21}u_{12} \ , \ U^\sim = \begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix} \ .$$

For the Koszul matrix $\Lambda_a = \Lambda_{(b,c)}$ of (6.3) we get

8.10 $$\Lambda_a = \begin{pmatrix} 0 & 0 & 0 \\ T & 0 & 0 \\ 0 & T^\sim & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & c & -b & 0 \end{pmatrix} \ .$$

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