MULTIVARIABLE OPERATOR THEORY ON DIRICHLET SPACES ON THE BALL

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The generalized Dirichlet space that $\mathcal{D}_q$ for each $q \in \mathbb{R}$ consists of holomorphic functions on the unit ball $\mathbb{B}$ of $\mathbb{C}^N$ and is a reproducing kernel Hilbert space. Reproducing kernels are $K_q(z,w) = (1 - \langle z, w\rangle)^{-(1+N+q)}$ for $q > -(N+1)$ and for the remaining q are the hypergeometric functions $K_q(z,w) = {}_2F_1(1,1;1-(N+q);\langle z,w\rangle)$. Here $\langle\cdot,\cdot\rangle$ is the natural inner product of $\mathbb{C}^N$.

If $q > -1$, these spaces are the standard weighted Bergman spaces A^2_q; for $q = -1$ the Hardy space H^2; for $q < -1$ the Besov spaces which we denote by B^2_q. Continuing, if $q = -(N+1)$, the reproducing kernel is logarithmic and $\mathcal{D}_{-(N+1)}$ is the classical Dirichlet space. Most importantly, if $q = -N$, $\mathcal{D}_{-N}$ is called the Drury-Arveson space which has a special place in operator theory.

On each $\mathcal{D}_q$ space, the operator tuple $S_q = (M_{z_1}, \ldots, M_{z_N})$ is called the multishift, where each term is the operator of multiplication by z_j for $z = (z_1, \ldots, z_N) \in \mathbb{B}$. If p is a polynomial and T is a contraction on a Hilbert space H, the one-variable von Neumann inequality says that $\|p(T)\| \leq \sup_{|z|<1} |p(z)|$.

The Drury-Arveson space $\mathcal{D}_{-N}$ was first used nontrivially by Drury (1978) who found the multivariable von Neumann inequality. This inequality has the form $\|p(T)\| \leq \|p(S_{-N})\|$ for a commuting N-tuple of contractions $T = (T_1, \ldots, T_N)$ and a polynomial p in N variables. Later Arveson (1998) investigated the various aspects of the space $\mathcal{D}_{-N}$ from the point of view of multivariable operator theory.

In this talk, we will see that many results known for $\mathcal{D}_{-N}$ ve S_{-N} can be extended to the whole Dirichlet-space family $\mathcal{D}_q$ and the operators S_q for $q \in \mathbb{R}$ by using a little more function theory. Radial derivatives of fractional order will play a key role in many places. Several of the results are valid also on weighted symmetric Fock spaces that are more general than the $\mathcal{D}_q$ spaces.

Every $\mathcal{D}_q$ space is a weighted symmetric Fock space as well as a function space defined by a Sobolev-type norm. There is a von Neumann inequality with respect to every $\mathcal{D}_q$ space. Explicitly, for a commuting tuple of contractions $T = (T_1, \ldots, T_N)$ and a polynomial of degree d in N variables, the inequality $\|p(T)\| \leq \sum_{k=0}^{d} \sqrt{K_k(q)}\, \|p_k(S_q)\|$ holds. Here p_k are the homogeneous terms of degree k of the polynomial p and the constants $K_k(q)$ are determined by $\mathcal{D}_q$. Continuing, the commutant of the operator S_q coincides with the multiplier algebra of the space $\mathcal{D}_q$. Further, the subnormal, hyponormal, essentially normal ranges of the operators S_q depend on q. Finally, the Toeplitz C^*-algebras generated by the S_q operators fit into short exact sequences.