# Evolution of roots of iterated derivatives of polynomials

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The Gauss-Lucas theorem says that the convex hull of the roots of the derivative of a polynomial is contained in the convex hull of the roots of the polynomial. Qualitatively, the convex hulls of the roots shrink under taking derivative. Can one make this quantitative? Given a polynomial $p$ , let $\sigma(p)$ denote the convex hull of the roots of $p$. A few weeks ago, I proved the following theorem that to my surprise turns out to be new. For any degree $n$ polynomial $p$ and any $c \geq \frac{1}{2}$,

$$\operatorname{Area} \sigma(p^{(cn)}) \leq 4(c - c^2) \operatorname{Area} \sigma(p).$$

Here $p^{(cn)}$ is shorthand for the $\lfloor cn \rfloor$'th derivative of $p$. This constant is independent of the polynomial $p$ or even the degree $n$.

Interestingly, the proof of the theorem is not particularly hard - The ingredients include a remarkable technique due to Joshua Batson, Adam Marcus, Daniel Spielman and Nikhil Srivastava which they call the barrier method and an easy to state and prove but powerful theorem concerning the relation between the roots of a polynomial and its derivative due to Rajesh Pereira and independently, Semyon Malamud. I'll give the proof of this quantitative Gauss-Lucas theorem and talk about the connections to geometric functional analysis. I'll also discuss a version of this result for random polynomials and tell you about some very interesting(at least to me) numerical simulations that I have made around this problem, which throw up plenty of questions.