ON THE PAPER "STATISTICAL APPROXIMATION BY POSITIVE LINEAR OPERATORS"

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Abstract. In this short paper, we show that the proof of the main result of the paper [1] is incorrect. In particular we note that the proof of the main results of [2], [3] and [4] are also incorrect. This remains that the main statements of those papers are conjectures.

1. The results of [1]

If f, $g : \mathbb{R} \to \mathbb{R}$ are two functions satisfying $f(x) \leqslant g(x)$ for all $x \in \mathbb{R}$ we write $f \leqslant g$. The function $|f| : \mathbb{R} \to \mathbb{R}$ is defined by $|f|(x) = |f(x)|$ for all $x \in \mathbb{R}$. A function $\rho : \mathbb{R} \to \mathbb{R}$ is called a *weight* if

$$\lim_{|x|\to\infty} \rho(x) = \infty \text{ and } \rho(x) \geqslant 1 \text{ for all } x \in \mathbb{R}.$$

We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *dominated* by ρ if there exists a positive real number $r \geqslant 0$ such that $|f| \leqslant r\rho$ The *weight space*, denoted by B_ρ, is the normed space of dominated functions by ρ with norm

$$||f||_\rho = \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\rho(x)}.$$

The subspace of B_ρ of continuous functions is denoted by C_ρ. Throughout the paper ρ_1 and ρ_2 will denote two weight functions satisfying

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0.$$

On can show that $\rho_1 \leqslant r\rho_2$ for some real number r, which implies that

$$C_{\rho_1} \subset C_{\rho_2} \text{ and } B_{\rho_1} \subset B_{\rho_2}.$$

A linear map T from C_{ρ_1} into B_{ρ_2} is called *positive* if $T(f) \geqslant 0$ whenever $f \geqslant \mathbf{0}$: here $\mathbf{0}$ stands for the constant zero function. It is well-known that a positive operator defined on a Banach lattice is bounded and the norm of a positive operator T from C_{ρ_1} into B_{ρ_2} is denoted by $||T||_{C_{\rho_1}\to B_{\rho_2}}$. That is,

$$||T||_{C_{\rho_1}\to B_{\rho_2}} = \sup_{||f||_{\rho_1}\leqslant 1} ||T(f)||_{\rho_2}.$$

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We fix the following notations:

- (A^n) stands for an infinite matrix with non-negative real entries satisfying

$$\sup_{n,k}\sum_{j=1}^{\infty} {a_{kj}}^{(n)} < \infty,$$

where

$$A^{(n)} = [a_{kj}^{(n)}].$$

- (L_j) denotes a sequence of positive operators from C_{ρ_1} into B_{ρ_2}.
- For $v = 0, 1, 2$, the function $F_v : \mathbb{R} \to \mathbb{R}$ is defined by

$$F_v(x) = \frac{x^v \rho_1(x)}{1+x^2}.$$

- For each $x \in \mathbb{R}$, the function $g_x \in C_{\rho_1}$ is defined by

$$g_x(t) = (t-x)^2 F_0(t).$$

- $B := \{f \in C_{\rho_1} : ||f||_{\rho_1} = 1\}$.

The following "theorem" is stated in [1] as the main result.

Theorem 1.1. *Suppose that for each $n \in \mathbb{N}$ and $v = 1, 2, 3$ we have*

$$\lim_k \sum_{j=1}^{\infty} {a_{kj}}^{(n)} ||L_j(F_v) - F_v||_{\rho_1} = 0 \text{ (uniformly)}$$

Then,

$$\lim_k \sum_{j=1}^{\infty} {a_{kj}}^{(n)} ||L_j(f) - f||_{\rho_2} = 0 \text{ (uniformly)}$$

for each $n \in \mathbb{N}$ and $f \in C_{\rho_1}$.

2. "Proof" of Theorem 1.1

In [1], the above "theorem" is proved using the following steps.

Step 1. (Lemma 1, [1]). Suppose that for any $0 \leqslant s \in \mathbb{R}$,

$$\lim_k \sum_{j=1}^{\infty} {a_{kj}}^{(n)} \sup_{f\in B} \sup_{|x|\leqslant s} \frac{|L_j(f)(x)|}{\rho_1(x)} = 0$$

and

$$\sup_{n,k} \sum_{j=1}^{\infty} {a_{kj}}^{(n)} ||L_j||_{C_{\rho_1}\to B_{\rho_1}} < \infty.$$

Then

$$\sup_{n,k} \sum_{j=1}^{\infty} {a_{kj}}^{(n)} ||L_j||_{C_{\rho_1}\to B_{\rho_2}} = 0.$$

Step 2. (Lemma 2, [1]). Suppose that

$$\sup_{n,k} \sum_{j=1}^{\infty} {a_{kj}}^{(n)} ||L_j||_{C_{\rho_1}\to B_{\rho_1}} < \infty,$$

and for each $n \in \mathbb{N}$ and $0 \leqslant s \in \mathbb{R}$, one has

$$\lim_k \sum_{j=1}^{\infty} {a_{kj}}^{(n)} \sup_{f\in B} \sup_{|x|\leqslant s} |L_j(f)(x) - f(x)| = 0.$$

Then, for each $f \in C_{\rho_1}$, the equality

$$\lim_k \sum_{j=1}^{\infty} a_{kj}{}^{(n)} ||L_j(f) - f||_{\rho_2} = 0$$

holds for all $n \in \mathbb{N}$.

Step 3. For each j, one has

$$||L_j||_{C_{\rho_1} \to B_{\rho_1}} \leqslant ||L_j(F_2) - F_2||_{\rho_1} + ||L_j(F_0) - F_0||_{\rho_1} + 1$$

Step 4. The following is true:

$$\begin{aligned} \sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^{(n)} ||L_j||_{C_{\rho_1} \to B_{\rho_1}} &\leqslant \sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^{(n)} ||L_j(F_2) - F_2||_{\rho_1} \\ &+ \sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^{(n)} ||L_j(F_0) - F_0||_{\rho_1} \\ &+ \sup_{n,k} \sum_{j=1}^{\infty} a_{kj}^{(n)} < \infty \end{aligned}$$

Step 5. Let $f \in C_{\rho_1}$ and $0 \leqslant s \in \mathbb{R}$ be given. For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon + K_{\rho_1}(x)(t-x)^2 F_0(t)$$

holds for all $t \in \mathbb{R}$ and $|x| \leqslant s$, where

$$K_{\rho_1}(x) = 4M_f \rho_1(x)\left(\frac{1+x^2}{\delta^2} + 1\right)$$

and M_f is a positive real number satisfying $|f| \leqslant M_f \rho$.

Step 6. Let $f \in C_{\rho_1}$ and $0 \leqslant s \in \mathbb{R}$ be given. For each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|L_j(f(t))(x) - f(x)| < \varepsilon L_j(1)(x) + K_{\rho_1}(x) L_j(g_x)(x) + |f(x)||L_j(1)(x) - 1|$$

for all $t \in \mathbb{R}$ and $|x| \leqslant s$, where $K_{\rho_1}(x)$ is as in step 5.

Step 7. Let $0 \leqslant s \in \mathbb{R}$ be given. For each $\varepsilon > 0$ we have

$$\begin{aligned} \sup_{f \in B} \sup_{|x| \leqslant s} |L_j(f(t))(x) - f(x)| &< C_1 \, \varepsilon \sup_{|x| \leqslant s} |L_j(1)(x)| + C_2 \sup_{|x| \leqslant s} L_j(g_x)(x) \\ &+ C_3 \sup_{|x| \leqslant s} ||L_j(1)(x) - 1| \end{aligned}$$

where,

- $C_1 = \sup_{|x| \leqslant s} \rho_1(x)$,
- $C_2 = \sup_{|x| \leqslant s} K_{\rho_1}(x)$, and
- $C_3 = \sup_{|x| \leqslant s} |f(x)|$.

3. Meaningless of the Theorem and Incorrectness of the "proof"

In this section we will explain why the above theorem is meaningless and show that even if the statement of the theorem is solidified using suitable conditions, its "proof" is still incorrect. First we note that since the operators in the sequence (L_j) are defined from C_{ρ_1} into B_{ρ_2}, all sentences involving the symbol " $||L_j||_{C_{\rho_1} \to B_{\rho_1}}$" are misleading as the range of L_i is not necessarily in B_{ρ_1}.

1. One of the problems in the statement of Theorem 1 of [1] is that although the operators L_j are defined from C_{ρ_1} into B_{ρ_2}, letting $||L_j||_{C_{\rho_1}\to B_{\rho_2}}$ means that one automatically supposes that $T_j(C_{\rho_1}) \subset B_{\rho_1}$, which is certainly not true.

2. In [1], in the proof of Step 2, it is supposed that the operators $T_j = L_j - I$, where I is identity operator, are positive. This is certainly not true.

3. Steps 3 and 4 are also meaningless as L_j takes its values in B_{ρ_2}, not in B_{ρ_1}.

4. Steps 5 and 6 are correct.

Now we can state the most serious mistake of the paper [1], which appears in Step 7, as follows:
5. **Step 7 is incorrect**: The equation of Step 7 follows from the equation of Step 6 by taking supreme over the set

$$B = \{f \in C_{\rho_1} : ||f||_{\rho_1} = 1\}.$$

We note that B is not equicontinuous, that is, there exists $\varepsilon > 0$ for which there is no $\delta > 0$ such that the following implication holds:

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| \leqslant \varepsilon \text{ for all } f \in B.$$

In Step 7, for each $0 \leqslant s \in \mathbb{R}$ and $f \in C_{\rho_1}$ the function

$$K_{\rho_1}(x) = 4M_f(x)(\tfrac{1+x^2}{\delta^2} + 1)$$

is defined. We must note that $\delta > 0$ in $K_{\rho_1}(x)$ depends on f. More precisely, one must have

$$K_{\rho_1}(x) = 4M_f(x)(\tfrac{1+x^2}{\delta_f{}^2} + 1).$$

With this in hand, in the equation of Step 7, C_2 must be in the form

$$C_2 = \sup_{||f||_{\rho_1}=1} \sup_{|x|\leqslant s} K_{\rho_1}(x).$$

But in this case, because B is not equicontinuous, we have

$$\sup_{f\in B} \tfrac{1}{\delta_f} = \infty.$$

Hence, in Step 7, $C_2 = \infty$, so nothing more can be performed in the proof.

All these are enough to show that the “proof” of the main result of [1] is incorrect.

4. Some related remarks

The ideas and the proof techniques of the papers [2], [3], and [4] are very similar (put differently, verbatim) to those of [1]. Although one can easily check that the operator "$L_j - I$" is not positive, in all these papers it is used as a positive operator. Unfourtunately, the above-mentioned incorrectness of the steps effects the proof of the main results of these papers as well. Hence the "proofs" of the main results of [2], [3] and [4] are also incorrect.

I don't know that either the main statement of the paper [1] is true or not. Hence, those statements are now conjectures.

References

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