Vanishing Theorems on a Compact Complex Manifold and Applications to the Hopf Conjecture and the (non-)Existence of Complex Structure on the 6-sphere

Cüneyt Ferahlar

Abstract

In his dissertation thesis [1], the author proved a Weitzenbock formula on complex manifolds, which involves the $\overline{\partial}-$Hodge Laplacian Δ_H, the Bochner Laplacian of the Levi-Civita connection Δ_R, and another Laplacian we construct, Δ_K, that is related to the Lefschetz operator and ∂ operator such that for any $(p,q)-$form, $\Delta_K + \Delta_H - 2\Delta_R = F(R)+$"1st order quadratic terms" where $F(R) : E^{p,q} \to E^{p,q}$ is a curvature operator [8], [6]. This generalizes a Weitzenbock formula of H. H. Wu on Kähler manifolds such that $\Delta_H - \Delta_R = F(R)$ in [11]. Under certain conditions, the author proves that this Weitzenbock formula provides vanishing theorems for the Dolbeault cohomology groups of complex differential $(p,q)-$forms and obtains information about the Hodge numbers of the manifold, $h^{p,q}$, in particular, the geometric genus, $\mathfrak{p}_g$, and arithmetic genus, $\mathfrak{p}_a$, and irregularity, $\mathfrak{q}$, of a compact complex manifold [7], [2]. Furthermore, the author uses the main vanishing theorem to obtain the Euler characteristic of the manifold $\chi(M)$ to show that the Hopf Conjecture [9] holds for a compact complex manifold with nonnegative sectional, holomorphic bisectional and isotropic curvature under certain extra conditions for $(p,q)-$forms [10], [3]. Finally, an earlier result of A. Gray states that a hypothetical integrable almost complex structure on the 6-sphere, S^6, has to satisfy $h^{0,1}\left(S^6\right) > 0$ [5]. Johnson and the author applies the main vanishing theorem in [4] for $(0,1)-$forms to show that $h^{0,1}\left(S^6\right) = 0$ and thus, under certain additional conditions S^6 can not admit an integrable almost complex structure.

References

[1] Ferahlar, C. *A Weitzenbock Formula for Compact Complex Manifolds and Applications to the Hopf Conjecture in Real Dimension 6*, Thesis Dissertation, Lehigh University, 2016, 168 pages; Proquest, publication number: 10125873

[2] Ferahlar, C. *Vanishing Theorems and Plurigenera of a compact complex manifold.* Preprint. 2018.

[3] Ferahlar, C. *Compact Complex manifolds and the Hopf Conjecture.* Preprint. 2018.

[4] Ferahlar, C., Johnson, D. L. *On the (non)-existence of an hypothetical integrable almost complex structure on the 6-sphere.* Preprint. 2018.

[5] Gray, A. *A Property of a Hypothetical complex structure on the six sphere.* Bolletino U.M.I., 1997, **3**, 251–255, MR1456264

[6] Griffiths, P., Harris, J. *Principles of algebraic geometry.* Reprint of the 1978 original. Wiley Classics Library. John Wiley & Sons, Inc., New York, (1994). xiv+813 pp. ISBN: 0-471-05059-8 14-01

[7] Hirzebruch, F. *Topological methods in algebraic geometry,* Third ed. Die Grundlehren der Mathematischen Wissenschaften, Band 131, Springer-Verlag New York, Inc., 1966. MR 0202713

[8] Hodge, W. V. D., *The Theory and Applications of Harmonic Integrals.* Cambridge University Press, 1941, MR 0003947.

[9] H. Hopf. *Differentialgeometrie und topologische Gestalt.* Jahresber. Deutsch. Math.-Verein, 41: 209–229, 1932.

[10] Wells, Jr., R. O., *Differential analysis on complex manifolds.* Graduate Texts in Mathematics, **Vol. 65**, 2nd Ed. Springer-Verlag, New York-Berlin, (1980). MR608414

[11] Wu, H. H. *The Bochner technique in differential geometry,* Math. Rep., Mathematical Reports, **Vol. 3**, 1988, 289–538, MR1079031