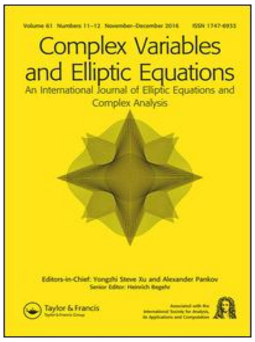

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On Hilbert-type boundary-value problem of poly-Hardy class on the unit disc†

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In this article, the poly-Hardy class on the unit disc is introduced and the boundary behaviour of the function in this class is discussed. Then the method used in Wang (Y.F. Wang, *On modified Hilbert boundary-value problems of polyanalytic functions*, Math. Methods Appl. Sci. 32 (2009), pp. 1415–1427) is applied to Hilbert-type boundary-value problems for the poly-Hardy class on the unit disc, and the expression of solution and the condition of solvability are explicitly obtained.

Keywords: poly-Hardy class; boundary behaviour; Hilbert-type problem; polyanalytic function

AMS Subject Classifications: 30E25; 30G30; 31A25; 45E05

1. Introduction

According to [1], a function f defined on the open set $G \subset \mathbb{C}$ is said to be a polyanalytic function of order n on G, or simply n-analytic function, if it satisfies the generalized Cauchy–Riemann equation $\partial_{\bar{z}}^n f(z) = 0$, $z \in G$, where $\partial_{\bar{z}} = 1/2[\partial/\partial x + i(\partial/\partial y)]$ is the classical Cauchy–Riemann operator. The generalized Cauchy–Riemann equation is often called the homogeneous polyanalytic equation, and hence the polyanalytic function is a solution of the homogeneous polyanalytic equation. As in [2], the class of all the polyanalytic functions on the open set G is denoted as $H_n(G)$, and $H_1(G)$ is just the set of analytic functions on G. An excellent overview of properties for the polyanalytic function is presented in Balk's monograph [1].

In general, differential equations coupling diverse boundary conditions come as various kinds of boundary-value problems (BVPs), such as Schwarz-type BVP, Hilbert-type BVP, Riemann-type BVP, Neumann-type BVP and Dirichlet-type BVP. The theory of BVPs for analytic functions and generalized analytic functions have already been presented in many references, e.g. [3–6]. Different BVPs of the polyanalytic function have been diffusely discussed, and the expression of solution and the condition of solvability are obtained, see e.g. [2,7–12]. BVPs for other types

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†Dedicated to Professor Okay Çelebi on the occasion of his 70th birthday.

of complex differential equations, including the nonhomogeneous polyanalytic equation [13,14], the meta-analytic equation [11,15,16], the polyharmonic equation [17], the higher order Poisson equation [18–20] and the nonhomogeneous Cauchy–Riemann equation [21], have been widely investigated in the recent years. The theory of BVPs is closely connected with that of singular integral equations, and it has application in physics [5]. In particular, BVPs for generalized higher order Poisson equations are transformed into equivalent singular integral equations by Aksoy and Çelebi [18,19].

Let $\mathbb{D}=\{z\colon |z|<1\}$ be the unit disc on the complex plane $\mathbb{C}$ and $\mathbb{T}=\{t\colon |t|=1\}$ be the unit circumference, oriented counterclockwise. A Hilbert-type BVP is to find a function $V\in H_n(\mathbb{D})$ such that $\partial_{\bar z}^k V$, $k=0,1,\ldots,n-1$ are continuous on the closed unit disc $\mathbb{D}\cup\mathbb{T}$, satisfying n Hilbert-type boundary conditions

$$\mathrm{Re}\Big\{[a_j(t)+ib_j(t)]\cdot\big[\partial_{\bar z}^k V\big]^+(t)\Big\}=c_j(t),\quad t\in\mathbb{T},\ j=0,1,\ldots,n-1 \tag{1.1}$$

with different factors $a_j,b_j,c_j\in H(\mathbb{T};\ \mathbb{R})$ (Hölder-continuous and real-valued) for $j=0,1,\ldots,n-1$. For the case of $n=2$, the Hilbert-type BVP (1.1) of bianalytic functions with the different factors has been discussed in detail [9]. For the case $a_j(t)+ib_j(t)\equiv 1$, $t\in\mathbb{T}$, $j=0,1,2,\ldots,n-1$, the Hilbert-type BVP (1.1) is reduced to the Schwarz problem of the polyanalytic function, which has been solved by Begehr and Schmersau [7]. If all the coefficients in the boundary conditions (1.1) also satisfy the proportion conditions $a_0(t)b_j(t)=a_j(t)b_0(t)$, $t\in\mathbb{T}$, $j=1,2,\ldots,n-1$, a class of modified Hilbert-type BVPs for polyanalytic functions has been discussed [8]. In addition, Hilbert-type BVPs or Schwarz-type BVPs on other special domains, such as the triangle, the upper half unit disc and the half ring, have been investigated, see for example [14,21,22].

In this article, the boundary conditions (1.1) are required to be satisfied almost everywhere on the unit circumference $\mathbb{T}$, i.e.,

$$\mathrm{Re}\Big\{[a(t)+ib(t)]\cdot\big[\partial_{\bar z}^k V\big]^+(t)\Big\}=c_j(t),\quad \text{a.e. } t\in\mathbb{T},\ \ j=0,1,\ldots,n-1, \tag{1.2}$$

where all the boundary data a, b, c_j are continuous on the unit circumference $\mathbb{T}$. In (1.1), $V\in H_n(\mathbb{D})\cap\{V\colon\ \partial_{\bar z}^k V\in C(\mathbb{T};\ \mathbb{R}),\ k=0,1,\ldots,n-1\}$, but V in (1.2) are required to belong to a broader class of functions, say the so-called poly-Hardy class on the unit disc. Then the boundary behaviour of the function in this class is discussed. Finally, in virtue of the decomposition of the poly-Hardy class, the explicit expression of solutions and the conditions of solvability are obtained.

2. Poly-Hardy class and its boundary behaviour

Let f be a function defined on the unit disc $\mathbb{D}$, and define

$$f_r(\theta)=f(re^{i\theta}),\quad 0\le r<1. \tag{2.1}$$

As in [23–25], the classical Hardy class on the unit disc is defined as

$$H^q(\mathbb{D})=\{f\in H_1(\mathbb{D})\colon\ \|f\|_{1,q}<+\infty\},$$

where

$$\|f\|_{1,q} = \sup\{\|f_r\|_q\colon\ 0 \le r < 1\} \tag{2.2}$$

with

$$\begin{cases} \|f_r\|_q = \left\{\dfrac{1}{2\pi}\displaystyle\int_{\mathbb{T}} |f_r(\theta)|^q \mathrm{d}\theta\right\}^{\frac{1}{q}}, & 0 < q < \infty, \\ \|f_r\|_\infty = \sup\big\{f_r(\theta)\colon\ \theta \in [0, 2\pi]\big\}. \end{cases} \tag{2.3}$$

When $q > 1$, the Hardy class $H^q(\mathbb{D})$ is a Banach space under the norm given in (2.2), see [23].

If $f \in H_n(\mathbb{D})$, $n > 1$, we define functions

$$f^j(z) = \partial_{\bar z}^j f(z),\ \ z \in \mathbb{D} \quad \text{for } j = 0, 1, 2, \ldots, n-1, \tag{2.4}$$

and hence $f^0(z) = f(z)$, $z \in \mathbb{D}$. Besides, $f \in H_n(\mathbb{D})$ admits the following classical decomposition [1,2]:

$$f(z) = f_0(z) + \bar z f_1(z) + \cdots + \bar z^{n-1} f_{n-1}(z), \quad z \in \mathbb{D},$$

where $f_j \in H_1(\mathbb{D})$ is j-component of f.

Definition 2.1 For $q > 0$ and $n > 1$, the subset of polyanalytic functions on $\mathbb{D}$

$$\{f \in H_n(\mathbb{D})\colon\ \|f^j\|_{1,q} < +\infty,\ \ j = 0, 1, \ldots, n-1\} \tag{2.5}$$

is called poly-Hardy class of order n on the unit disc, where $\|\cdot\|_{1,q}$ is defined as (2.2) and f^j given in (2.4). Such a poly-Hardy class on the unit disc is denoted as $H_n^q(\mathbb{D})$.

Obviously, $H_n^q(\mathbb{D})$, $q > 0$ is a linear space. Let

$$\|f\|_{n,q} = \sum_{j=0}^{n-1} \|f^j\|_{1,q} \quad \forall f \in H_n^q(\mathbb{D}), \tag{2.6}$$

where $\|\cdot\|_{1,q}$ is given in (2.2). In what follows, we always assume $H_1^q(\mathbb{D}) = H^q(\mathbb{D})$. If $n = 1$, the symbol $\|\cdot\|_{n,q}$ defined as (2.6) is just $\|\cdot\|_{1,q}$ given in (2.2). Thus, by Definition 2.1, one has $H_n^q(\mathbb{D}) = \{f \in H_n(\mathbb{D})\colon\ \|f\|_{n,q} < +\infty\}$. Now we have the following decomposition of the poly-Hardy class.

THEOREM 2.1 *If $H_n^q(\mathbb{D})$, $q > 0$ is the poly-Hardy class of order n $(n > 1)$ on the unit disc, then $H_n^q(\mathbb{D}) = H_1^q(\mathbb{D}) \oplus \bar z H_1^q(\mathbb{D}) \oplus \cdots \oplus \bar z^{n-1} H_1^q(\mathbb{D})$, where $\bar z^j H_1^q(\mathbb{D}) = \{\bar z^j f(z)\colon f \in H_1^q(\mathbb{D})\}$ for $j = 0, 1, \ldots, n-1$.*

Proof First, one needs to prove

$$H_n^q(\mathbb{D}) \subset H_1^q(\mathbb{D}) + \bar z H_1^q(\mathbb{D}) + \cdots + \bar z^{n-1} H_1^q(\mathbb{D}). \tag{2.7}$$

In fact, if $f \in H_n^q(\mathbb{D})$, then $f \in H_n(\mathbb{D})$ by Definition 2.1, and hence $f(z) = f_0(z) + \bar z f_1(z) + \cdots + \bar z^{n-1} f_{n-1}(z)$ with $f_j \in H_1(\mathbb{D})$ for $j = 0, 1, \ldots, n-1$. By the simple computation, one immediately gets the expression in the matrix form

$$\mathbf{f}^{\#}(z) = \mathbf{A}(\bar z)\mathbf{f}_{\#}(z), \quad z \in \mathbb{D}, \tag{2.8}$$

with

$$\mathbf{f}^{\#}(z)=\begin{pmatrix} f^0(z)\\ f^1(z)\\ \vdots\\ f^{n-1}(z)\end{pmatrix},\quad \mathbf{f}_{\#}(z)=\begin{pmatrix} f_0(z)\\ f_1(z)\\ \vdots\\ f_{n-1}(z)\end{pmatrix}\tag{2.9}$$

and

$$\mathbf{A}(z)=\begin{pmatrix} 1 & z & z^2 & \cdots & z^{n-1}\\ 0 & 1! & 2z & \cdots & (n-1)z^{n-2}\\ 0 & 0 & 2! & \cdots & (n-1)(n-2)z^{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & (n-1)!\end{pmatrix}.\tag{2.10}$$

In (2.9), f^j, $j=0, 1, \ldots, n-1$ are defined by (2.4) and f_j is j-component of f. (2.8) directly leads to

$$\mathbf{f}_{\#}(z)=\mathbf{A}^{-1}(\bar{z})\mathbf{f}^{\#}(z),\quad z\in\mathbb{D},\tag{2.11}$$

where $\mathbf{A}^{-1}(z)$ is the inverse of $\mathbf{A}(z)$ given in [10], i.e., $\mathbf{A}^{-1}(z)=(b_{kj}(z))_{n\times n}$ with

$$b_{kj}(z)=\begin{cases}\dfrac{(-1)^{k+j}}{(j-k)!(k-1)!}z^{j-k}, & j\geq k,\\ 0, & j<k.\end{cases}$$

Therefore, (2.11) implies $f_j\in H_1^q(\mathbb{D})$ for $j=0, 1, \ldots, n-1$, which leads to the validity of (2.7). Hence one has $H_n^q(\mathbb{D})=H_1^q(\mathbb{D})+\bar{z}H_1^q(\mathbb{D})+\cdots+\bar{z}^{n-1}H_1^q(\mathbb{D})$.

Finally, if $0=f_0(z)+\bar{z}f_1(z)+\cdots+\bar{z}^{n-1}f_{n-1}(z)$ with $f_j\in H_1^q(\mathbb{D})$ for $j=0, 1, \ldots, n-1$, it follows from (2.11) that $f_j(z)\equiv 0$, $z\in\mathbb{D}$ for $j=0, 1, \ldots, n-1$. This completes the proof. ■

For an arbitrary function in the poly-Hardy class $H_n^q(\mathbb{D})$, the following boundary behaviour is true.

Theorem 2.2 *If $f\in H_n^q(\mathbb{D})$, $q>0$, then f has the nontangential limits $f^+(t)$ almost everywhere on $\mathbb{T}$, and*

$$\lim_{r\to 1}\|f^+-f_r\|_q=0,\tag{2.12}$$

where f_r, $\|\cdot\|_q$ are given in (2.1) *and* (2.3), *respectively.*

Proof If $f\in H_n^q(\mathbb{D})$, $q>0$, by Theorem 2.1, one has

$$f(z)=f_0(z)+\bar{z}f_1(z)+\cdots+\bar{z}^{n-1}f_{n-1}(z),\tag{2.13}$$

with $f_j(z)\in H_1^q(\mathbb{D})$ for $j=0, 1, \ldots, n-1$. By the boundary behaviour of the function in the classical Hardy class $H_1^q(\mathbb{D})$ (see Theorem 17.12 in [23]), f_j has the nontangential limits $f_j^+(t)$ almost everywhere on $\mathbb{T}$, and

$$\lim_{r\to 1}\left\| f_j^+ - (f_j)_r \right\|_q = 0, \tag{2.14}$$

where $(f_j)_r$ is defined in (2.1). Let

$$f^+(t) = f_0^+(t) + \bar{t}f_1^+(t) + \cdots + \bar{t}^{n-1}f_{n-1}^+(t), \quad \text{a.e. } t\in\mathbb{T}. \tag{2.15}$$

Therefore, the decomposition (2.13) implies that f has the nontangential limits $f^+(t)$ defined as (2.15) almost everywhere on $\mathbb{T}$. By (2.13) and (2.15),

$$\|f^+ - f_r\|_q \le \sum_{j=0}^{n-1}\left[\|f_j^+ - (f_j)_r\|_q + (1-r^j)\cdot\|(f_j)_r\|_q\right],$$

which leads to the validity of (2.12) by virtue of (2.14). The proof is completed. ∎

Remark 2.1 In Theorem 2.2, f^+ is usually called the nontangential boundary-value of the function f. In what follows, the symbol f^+ is always understood as the nontangential boundary value of f.

By Theorem 2.1, $f\in H_n^q(\mathbb{D})$ also admits the following decomposition:

$$f(z) = f_0(z) + (\bar{z}+z)f_1(z) + \cdots + (\bar{z}+z)^{n-1}f_{n-1}(z), \quad z\in\mathbb{D}, \tag{2.16}$$

with $f_j(z)\in H_1^q(\mathbb{D})$ for $j=0,\ 1,\ldots,n-1$. The decomposition (2.16) will be used in Section 4.

3. Hilbert BVP for Hardy class

In this section, we introduce the theory of Hilbert-type BVP for the Hardy class on the unit disc under the continuous boundary data. For the case of the piecewise continuous boundary data, we refer readers to reference [24].

Let $C(\mathbb{T};\mathbb{R})$ denote the set of all the real-valued continuous functions defined on the unit circumference $\mathbb{T}$. Our problem is to find a function $\Phi\in H^q(\mathbb{D})$, $q>1$ satisfying the following Hilbert-type boundary condition

$$\mathrm{Re}\left\{[a(t)+ib(t)]\Phi^+(t)\right\} = c(t), \quad \text{a.e. } t\in\mathbb{T}, \tag{3.1}$$

where $a, b, c\in C(\mathbb{T};\mathbb{R})$ and $a^2(t)+b^2(t)\equiv 1$, $t\in\mathbb{T}$. The problem (3.1) is called Hilbert BVP of the Hardy class.

One of the simple Hilbert-type BVP for the Hardy class is the nonhomogeneous Schwarz problem, i.e., $a(t)+ib(t)\equiv 1$, $t\in\mathbb{T}$ in (3.1). The following lemma is the classical result, and readers can refer reference [24].

LEMMA 3.1 *The nonhomogeneous Schwarz problem*

$$\begin{cases} \Phi\in H^q(\mathbb{D}), & q>1, \\ \mathrm{Re}\{\Phi^+(t)\} = c(t), & \text{a.e. } t\in\mathbb{T} \text{ with } c\in L_q(\mathbb{T};\mathbb{R}), \\ \mathrm{Im}\{\Phi(0)\} = 0 & \end{cases}$$

has the unique solution

$$\Phi(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} c(\tau)\frac{\tau + z}{\tau - z}\frac{d\tau}{\tau}, \quad z \in \mathbb{D}.$$

The following conclusion, due to V.I. Smirnov, is often called Smirnov's theorem, which can be found in [24,25].

LEMMA 3.2 *Suppose* $h(z) = \varphi(z) + i\psi(z) \in H_1(\mathbb{D})$, *where* φ, ψ *are the real part and the imaginary part, respectively. If* $\varphi \in C(\mathbb{D} \cup \mathbb{T})$, *then* $\exp\{ih(z)\} \in H^q(\mathbb{D})$ *for any* $q > 0$.

Let

$$\kappa = \frac{1}{2\pi}\{\arg[a(t) - ib(t)]\}_{\mathbb{T}}, \tag{3.2}$$

which is often said to be the index of Hilbert problem (3.1), see [4,8,9]. Introduce the Schwarz operator S as in [8,9], i.e.,

$$S[\alpha](z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \alpha(\tau)\frac{\tau + z}{\tau - z}\frac{d\tau}{\tau}, \quad z \in \mathbb{D}, \tag{3.3}$$

with

$$\alpha(t) = \arg\{t^{-\kappa}[a(t) - ib(t)]\}, \quad t \in \mathbb{T}. \tag{3.4}$$

Now define

$$X(z) = iz^{\kappa}\exp\{iS[\alpha](z)\}, \quad z \in \mathbb{D}\backslash\{0\}, \tag{3.5}$$

where S is just the Schwarz operator. And we have the following result, which generalizes the corresponding conclusion in [8,9].

THEOREM 3.1 *Let X be defined by* (3.5). *Then X has the following properties*:

(1) $\mathrm{Re}\{[a(t) + ib(t)]X^+(t)\} = 0$, *a.e.* $t \in \mathbb{T}$.
(2) *For any* $q > 0$, $z^{-\kappa}X(z) \in H^q(\mathbb{D})$, $[z^{-\kappa}X(z)]^{-1} \in H^q(\mathbb{D})$.
(3) $X(z) \neq 0$, $z \in \mathbb{D}\backslash\{0\}$ *and* $X^+(t) \neq 0$, *a.e.* $t \in \mathbb{T}$.
(4) $X(z)$ *possesses a pole of order* $-\kappa$ *at* $z = 0$. *In the precise words,* $\lim_{z\to 0} z^{-\kappa}X(z) = c \neq 0$, *where c is a complex constant.*

Such a function X is called the canonical function of Hilbert problem (3.1). *Further, if* X_1, X_2 *are two canonical functions of Hilbert problem* (3.1), *there exists a constant* $c \in \mathbb{C}$, $c \neq 0$, *such that* $X_1(z) = c \cdot X_2(z)$, $z \in \mathbb{D}$.

Proof Let X be defined by (3.5) and $S[\alpha]$ be denoted as $u + iv$, i.e., $S[\alpha](z) = u(z) + iv(z)$, $z \in \mathbb{D}$. Since $\alpha \in C(\mathbb{T}; \mathbb{R})$, one has $u^+(t) = \alpha(t)$, $t \in \mathbb{T}$ by Lemma 3.1. By the relation between the harmonic function and its conjugate (see §2 of Chapter 9 in [25]), $v^+(t)$ exists almost everywhere on the unit circumference $\mathbb{T}$. Hence

$$X^+(t) = it^{\kappa}\exp\{i\alpha(t) - v^+(t)\}, \quad \text{a.e. } t \in \mathbb{T}, \tag{3.6}$$

which in turn implies

$$\mathrm{Re}\Big\{[a(t) + ib(t)]X^+(t)\Big\} = \mathrm{Re}\Big\{it^{\kappa}[a(t) + ib(t)]t^{-\kappa}[a(t) - ib(t)]e^{-v^+(t)}\Big\} = 0 \quad \text{a.e. } t \in \mathbb{T}. \tag{3.7}$$

The first equality in (3.7) follows from

$$e^{i\alpha(t)} = t^{-\kappa}[a(t) - ib(t)], \quad t \in \mathbb{T}.$$

Therefore, Property 1 remains true.

Property 2 follows directly from Lemma 3.2. By (3.5) and (3.6), Property 3 is also valid. Since $\lim_{z\to 0} z^{-\kappa}X(z) = i\exp\{iS[\alpha](0)\} \neq 0$, Property 4 is obvious.

Finally, one needs to prove that, if X_1, X_2 are two canonical functions of Hilbert problem (3.1), there exists a constant $c \in \mathbb{C}$, $c \neq 0$, such that $X_1(z) = c \cdot X_2(z)$, $z \in \mathbb{D}$. As a matter of fact, Property 1 implies

$$\operatorname{Re}\Big\{[a(t) + ib(t)]X_j^+(t)\Big\} = 0, \quad \text{a.e. } t \in \mathbb{T} \text{ for } j = 1, 2. \tag{3.8}$$

By Property 3, (3.8) leads to

$$\operatorname{Re}\big\{F^+(t)\big\} = 0, \quad \text{a.e. } t \in \mathbb{T}, \tag{3.9}$$

with

$$F(z) = \frac{X_1(z)}{iX_2(z)}, \quad z \in \mathbb{D}\backslash\{0\}.$$

On the other hand, Property 4 implies

$$\lim_{z\to 0} F(z) = -ic, \quad c \neq 0. \tag{3.10}$$

By Property 2 and the Hölder inequality, one gets

$$F \in H^q(\mathbb{D}), \quad q > 1. \tag{3.11}$$

Combining (3.9), (3.10) and (3.11), one immediately gets a Schwarz-type problem. By Lemma 3.1, $F(z) = -ic$, $z \in \mathbb{D}$, and hence $X_1(z) = c \cdot X_2(z)$, $z \in \mathbb{D}$. This completes the proof of the theorem. ■

Remark 3.1 By the boundary behaviour of the function in the Hardy class $H_1^q(\mathbb{D})$ [23,24], Theorem 3.1 implies $X^+ \in L_q(\mathbb{T})$ for any $q > 0$.

Next, we will discuss the homogeneous Hilbert BVP (3.1): $c(t) \equiv 0$, $t \in \mathbb{T}$ in (3.1), i.e.,

$$\operatorname{Re}\big\{[a(t) + ib(t)]\Phi^+(t)\big\} = 0, \quad \text{a.e. } t \in \mathbb{T}. \tag{3.12}$$

The class of symmetric Laurent polynomials is defined as in [8,9]

$$S\Pi_k = \left\{\pi_k = \sum_{j=-k}^{k} c_j z^j : \operatorname{Re} c_0 = 0, c_j = -\overline{c_{-j}},\ j = 1, 2, \ldots, k\right\}, \quad k \geq 0.$$

If $k < 0$, we assume $S\Pi_k = \{0\}$. Obviously,

$$\operatorname{Re}\{\pi_k(t)\} = 0, \quad t \in \mathbb{T},$$

for any $\pi_k \in S\Pi_k$. Similar to the discussion in [9], the solution of the homogeneous Hilbert-type BVP (3.12) is obtained.

Lemma 3.3 *The homogeneous Hilbert BVP* (3.12) *for the Hardy class is solvable and its solution can be written as*

$$\Phi(z) = X(z)\cdot\pi_\kappa(z) \quad \textit{with } \pi_\kappa \in S\Pi_\kappa,$$

where κ, X *are defined by* (3.2) *and* (3.5), *respectively*.

Proof For the moment, let

$$H_{1,\ell}(\mathbb{D}) = \{\varphi \in H_1(\mathbb{D}\backslash\{0\})\colon\ \mathrm{Ord}(\varphi, 0) \le \ell\}, \quad \ell \in \mathbb{Z},$$

where the symbol Ord(φ, 0) is the order of φ at $z=0$ defined as in [8]. The operator L_ℓ: $H_{1,\ell}(\mathbb{D}) \to S\Pi_\ell$ is introduced as follows [9]:

$$L_\ell\,[\varphi](z) = \sum_{j=-\ell}^{\ell} c_j z^j \tag{3.13}$$

with

$$c_j = \begin{cases} -\overline{a_{-j}}, & j>0,\\ i\,\mathrm{Im}\,a_0, & j=0,\\ a_j, & j<0, \end{cases}$$

where a_j is the coefficient of Laurent's expansion of $\varphi \in H_{1,\ell}(\mathbb{D})$ at the origin given by

$$\varphi(z) = \sum_{j=-\ell}^{\infty} a_j z^j, \quad z \in \mathbb{D}\backslash\{0\}.$$

By Theorem 3.1, (3.12) is equivalent to the following Schwarz-type BVP:

$$\begin{cases} \mathrm{Re}\{F^+(t)\} = 0, & \text{a.e. } t \in \mathbb{T},\\ \mathrm{Im}\{F(0)\} = 0, \end{cases} \tag{3.14}$$

with $\varphi(z) = \Phi(z)\cdot[iX(z)]^{-1} \in H_1^q(\mathbb{D})$ and $F(z) = \varphi(z) - L_\kappa[\varphi](z) \in H_1^q(\mathbb{D})$, where κ, X are defined by (3.2) and (3.5), respectively. By Lemma 3.1, one has $F(z)\equiv 0$, $z \in \mathbb{D}$, which leads to the validity of Lemma 3.3. ■

For the nonhomogeneous Hilbert-type BVP (3.1), one easily gets the following result by Lemmas 3.1 and 3.3 [9,24].

Theorem 3.2 *When* $\kappa \ge 0$, *Hilbert BVP* (3.1) *for the Hardy class is solvable and its solution can be written as*

$$\Phi(z) = \frac{X(z)}{2\pi i}\int_{\mathbb{T}} \frac{c(\tau)}{[a(\tau)+ib(\tau)]X^+(\tau)}\frac{\tau+z}{\tau-z}\frac{\mathrm{d}\tau}{\tau} + iX(z)\cdot\pi_\kappa(z) \quad \textit{with} \quad \pi_\kappa(z) \in S\Pi_\kappa.$$

When $\kappa < 0$, *if and only if the conditions of solvability*

$$\int_{\mathbb{T}} \frac{c(\tau)}{[a(\tau)+ib(\tau)]X^+(\tau)}\tau^{-\ell-1}\mathrm{d}\tau = 0, \quad \ell = 0,1,\ldots,-\kappa-1$$

are satisfied, Hilbert BVP (3.1) *for the Hardy class is solvable and its solution can be expressed as*

$$\Phi(z) = \frac{\exp\{\Gamma(z)\}}{2\pi i}\int_{\mathbb{T}} \frac{c(\tau)}{[a(\tau)+ib(\tau)]\exp\{\Gamma^+(\tau)\}}\frac{\mathrm{d}\tau}{\tau - z},$$

with

$$\Gamma(z) = \frac{1}{4\pi i}\int_{\mathbb{T}}\left[\log\left(-\tau^{-2\kappa}\frac{a(\tau)-ib(\tau)}{a(\tau)+ib(\tau)}\right)\right]\frac{\tau+z}{\tau-z}\frac{\mathrm{d}\tau}{\tau}.$$

4. Hilbert BVP for poly-Hardy class

The simplest Hilbert problem for the poly-Hardy class is to find a function $V \in H_n^q(\mathbb{D})$, $q > 1$ satisfying the following boundary conditions:

$$\mathrm{Re}\left\{\left[\partial_{\bar z}^j V\right]^+(t)\right\} = 0, \quad \text{a.e. } t\in\mathbb{T} \text{ for } j = 0,1,\ldots,n-1. \tag{4.1}$$

This problem is called the homogeneous Schwarz BVP of the poly-Hardy class on the unit disc. As in [8], let

$$\mathfrak{S}_m = \begin{cases} S\Pi_m \oplus (\bar z + z)S\Pi_m \oplus \cdots \oplus (\bar z + z)^{n-1}S\Pi_m, & m \geq 0,\\ \{0\}, & m < 0,\end{cases} \tag{4.2}$$

and one gets the following result.

THEOREM 4.1 *The homogeneous Schwarz BVP* (4.1) *for the poly-Hardy class is solvable and the set of solutions is* $\mathfrak{S}_0$, *where* $\mathfrak{S}_0$ *is defined as* (4.2).

Proof By the decomposition (2.16),

$$V(z) = V_0(z) + (\bar z + z)V_1(z) + \cdots + (\bar z + z)^{n-1}V_{n-1}(z) \quad \text{with } V_j \in H_1^q(\mathbb{D}),$$

for $j = 0,1,\ldots,n-1$. Thus, substituting this decomposition into the boundary conditions (4.1), one easily gets n independent Schwarz BVPs of the Hardy class on the unit disc

$$\mathrm{Re}\left\{V_j^+(t)\right\} = 0, \quad \text{a.e. } t\in\mathbb{T},\ j = 0,1,\ldots,n-1,$$

and hence $V_j \in S\Pi_0$ for $j = 0,1,\ldots,n-1$ by Lemma 3.1. This completes the proof. ■

The homogeneous Hilbert-type BVP for the poly-Hardy class is to find $V \in H_n^q(\mathbb{D})$ satisfying

$$\mathrm{Re}\left\{[a(t)+ib(t)]\cdot\left[\partial_{\bar z}^j V\right]^+(t)\right\} = 0, \quad \text{a.e. } t\in\mathbb{T} \text{ for } j = 0,1,\ldots,n-1, \tag{4.3}$$

where $a, b \in C(\mathbb{T};\mathbb{R})$.

THEOREM 4.2 *The homogeneous Hilbert-type BVP* (4.3) *for the poly-Hardy class is solvable and its solution can be represented as*

$$V(z) = iX(z)\pi_\kappa(z) \quad \textit{with } \pi_\kappa \in \mathfrak{S}_\kappa, \tag{4.4}$$

where X is defined as (3.5). *Namely, the set of solutions of the homogeneous Hilbert-type BVP* (4.3) *is* $iX\mathfrak{S}_\kappa$.

Proof By Properties 1, 2 and 3 of the canonical function X in Theorem 3.1, the boundary conditions (4.3) are equivalent to

$$\mathrm{Re}\left\{\left[\partial_{\bar z}^j W\right]^+(t)\right\} = 0, \quad \text{a.e. } t \in \mathbb{T} \quad \text{for } j = 0, 1, \ldots, n-1, \tag{4.5}$$

with

$$W(z) = \frac{V(z)}{iX(z)}, \quad z \in \mathbb{D}\backslash\{0\}.$$

Let $W(z) = W_0(z) + (\bar z + z)W_1(z) + \cdots + (\bar z + z)^{n-1} W_{n-1}(z)$ with $\mathrm{Ord}(W_j, 0) \le \kappa$ for $j = 0, 1, \ldots, n-1$. Since (4.5) is equivalent to $\mathrm{Re}\{\varphi_j^+(t)\} = 0$, a.e. $t \in \mathbb{T}$ with $\varphi_j = W_j - L_\kappa[W_j] \in H_1^q(\mathbb{D})$ for $j = 0, 1, \ldots, n-1$, where the operator L_κ is given in (3.13), one immediately gets $\varphi_j(z) = 0$, $z \in \mathbb{D}$ by Lemma 3.1. This leads to the validity of (4.4). ■

Finally, we discuss the nonhomogeneous Hilbert-type BVP for the poly-Hardy class: find a function $V \in H_n^q(\mathbb{D})$ satisfying the following Hilbert-type conditions:

$$\mathrm{Re}\left\{[a(t) + ib(t)] \cdot \left[\partial_{\bar z}^k V\right]^+(t)\right\} = c_j(t), \quad \text{a.e. } t \in \mathbb{T} \text{ for } j = 0, 1, \ldots, n-1, \tag{4.6}$$

where $a, b, c_j \in C(\mathbb{T}; \mathbb{R})$, $j = 0, 1, \ldots, n-1$.

As in [8], the poly-Schwarz operator on the unit circumference $\mathbb{T}$ is introduced as follows:

$$S[\gamma_0, \ldots, \gamma_{n-1}](z) = \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \cdot \frac{1}{2\pi i} \int_{\mathbb{T}} \gamma_k(\tau)(\tau - z + \overline{\tau - z})^k \frac{\tau + z}{\tau - z} \frac{\mathrm{d}\tau}{\tau}, \quad z \in \mathbb{D}, \tag{4.7}$$

where the kernel functions $\gamma_k \in L_q(\mathbb{T}; \mathbb{R})$ for $k = 0, 1, \ldots, n-1$. By Lemma 3.1, one immediately has

$$\mathrm{Re}\left\{\left[\partial_{\bar z}^j S[\gamma_0, \ldots, \gamma_{n-1}]\right]^+(t)\right\} = \gamma_j(t), \quad \text{a.e. } t \in \mathbb{T} \text{ for } j = 0, 1, \ldots, n-1. \tag{4.8}$$

By Theorem 3.1, boundary conditions (4.6) are transformed to

$$\mathrm{Re}\left\{\left[\partial_{\bar z}^j\left(\frac{V}{iX}\right)\right]^+(t)\right\} = \frac{c_j(t)}{i[a(t) + ib(t)]X^+(t)}, \quad \text{a.e. } t \in \mathbb{T} \text{ for } j = 0, 1, \ldots, n-1, \tag{4.9}$$

where the canonical function X is defined by (3.5). By Remark 3.1,

$$\frac{c_j(t)}{i[a(t) + ib(t)]X^+(t)} \in L_q(\mathbb{D}; \mathbb{R}) \quad \text{for } j = 0, 1, \ldots, n-1,$$

and hence the function

$$U(z) = X(z)\cdot S\left[\frac{c_0}{(a+ib)X^+},\ldots,\frac{c_{n-1}}{(a+ib)X^+}\right](z) \tag{4.10}$$

satisfies the boundary conditions (4.9) in virtue of (4.8). Therefore, we have

$$\mathrm{Re}\left\{\left[\partial_{\bar z}^{j}\left(\frac{V-U}{iX}\right)\right]^{+}(t)\right\} = 0, \quad \text{a.e. } t\in\mathbb{T} \text{ for } j=0,1,\ldots,n-1. \tag{4.11}$$

Now the Hilbert-type BVP (4.6) for the poly-Hardy class is discussed in two cases.

Case 1 When $\kappa \geq 0$, by Theorem 4.2, the solution of the Hilbert-type BVP (4.6) for the poly-Hardy class can be represented as

$$V(z) = U(z) + iX(z)\pi_\kappa(z) \quad \text{with } \pi_\kappa \in \mathfrak{S}_\kappa. \tag{4.12}$$

Case 2 When $\kappa < 0$, if the Hilbert-type problem (4.6) for the poly-Hardy class is solvable, then (4.11) leads to

$$\frac{V-U}{iX} \in \mathfrak{S}_0.$$

Since U given in (4.10) can be decomposed as

$$U(z) = X(z)\sum_{j=0}^{n-1}(\bar z + z)^j(-1)^j U_j(z), \tag{4.13}$$

with

$$U_j(z) = \sum_{k=j}^{n-1}\frac{(-1)^k}{j!(k-j)}\cdot\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{c_k(\tau)(\bar\tau+\tau)^{k-j}}{[a(\tau)+ib(\tau)]X^+(\tau)}\frac{\tau+z}{\tau-z}\frac{d\tau}{\tau} \quad \text{for } j=0,1,\ldots,n-1, \tag{4.14}$$

one has

$$V(z) = X(z)\sum_{j=0}^{n-1}(\bar z+z)^j(-1)^j\left[U_j(z)-c_j\right] \quad \text{with } c_j\in\mathbb{R} \tag{4.15}$$

for $j=0,1,\ldots,n-1$. Obviously, $V(z)$ given in (4.15) is the solution of the Hilbert-type BVP (4.6) for the poly-Hardy class if and only if the limit $\lim_{z\to 0} z^\kappa[U_j(z)-c_j]$ exists for $j=0,1,\ldots,n-1$. At the neighbourhood of the origin,

$$\frac{\tau+z}{\tau-z} = 1 + 2\sum_{\ell=1}^{+\infty}\left(\frac{z}{\tau}\right)^\ell = 1 + 2\sum_{\ell=1}^{-\kappa-1}\left(\frac{z}{\tau}\right)^\ell + \frac{2z^{-\kappa}\tau^{\kappa+1}}{\tau-z},$$

one easily gets $c_j = 0$ for $j=0,\ 1,\ldots,n-1$, and

$$\sum_{k=j}^{n-1}\frac{(-1)^k}{j!(k-j)!}\int_{\mathbb{T}}\frac{c_k(\tau)(\bar\tau+\tau)^{k-j}}{[a(\tau)+ib(\tau)]X^+(\tau)}\frac{d\tau}{\tau^{\ell+1}} = 0 \quad \text{for } \begin{cases}\ell = 0,1,\ldots,-\kappa-1,\\ j=0,1,\ldots,n-1.\end{cases} \tag{4.16}$$

Therefore, if the conditions (4.16) are fulfilled, the solution of the Hilbert BVP (4.6) for the poly-Hardy class can be rewritten as

$$V(z) = \sum_{j=0}^{n-1} (\bar{z} + z)^j (-1)^j V_j(z), \tag{4.17}$$

with

$$V_j(z) = \sum_{k=j}^{n-1} \frac{(-1)^k}{j!(k-j)!} \cdot \frac{\exp\{iS[\alpha](z)\}}{\pi i} \int_{\mathbb{T}} \frac{c_k(\tau)(\bar{\tau} + \tau)^{k-j}}{[a(\tau) + ib(\tau)] \exp\{iS[\alpha]^+(\tau)\}} \frac{\tau \mathrm{d}\tau}{\tau - z} \quad \text{for } j = 0, 1, \dots, n-1, \tag{4.18}$$

where $S[\alpha]^+(\tau)$ is the nontangential boundary value of $S[\alpha]$ given in (3.3). To sum up the discussion above, the main result is obtained.

THEOREM 4.3 *When $\kappa \geq 0$, the solution of the nonhomogeneous Hilbert-type BVP* (4.6) *for the poly-Hardy class can be represented as* (4.12). *When $\kappa < 0$, if and only if the conditions* (4.16) *are fulfilled, the nonhomogeneous Hilbert-type BVP* (4.6) *for the poly-Hardy class is solvable and its solution can be written as* (4.17) *with* (4.18).

Remark 4.1 Similar to the discussion in [8], the results obtained here can be generalized to the modified Hilbert-type BVP for the poly-Hardy class on the unit disc. In particular, if $c_j(t) = c(t)$, $t \in \mathbb{T}$ for $j = 0, 1, \dots, n-1$, then the conditions of solvability (4.16) can be reduced to

$$\int_{\mathbb{T}} \frac{c(\tau)(\bar{\tau} + \tau)^j}{[a(\tau) + ib(\tau)]X^+(\tau)} \frac{\mathrm{d}\tau}{\tau^{\ell+1}} = 0 \quad \text{for } \begin{cases} \ell = 0, 1, \dots, -\kappa - 1, \\ j = 0, 1, \dots, n-1. \end{cases}$$

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