Jensen-Pólya Program for the Riemann Hypothesis and Related Problems

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Joint with Michael Griffin, Larry Rolen, and Don Zagier

Riemann's zeta-function

Definition (Riemann)

For $\mathrm{Re}(s) > 1$, define the **zeta-function** by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

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1. *The function $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ (apart from a simple pole at $s = 1$ with residue 1).*
2. *We have the* **functional equation**

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \cdot \zeta(1-s).$$

Hilbert's 8th Problem

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Apart from the negative evens, the zeros of $\zeta(s)$ *satisfy* $\mathrm{Re}(s) = \frac{1}{2}$.

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"Without doubt, it would be desirable to have a rigorous proof of this proposition; however, I have left this research...because it appears to be unnecessary for the immediate goal of my study...."

Bernhard Riemann (1859)

Important Remarks

Huge Understatement

A proof of RH would clarify our understanding of primes.

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A proof of RH would clarify our understanding of primes.

What was known?

1. *The first "gazillion" zeros satisfy RH (van de Lune, Odlyzko).*
2. $>41\%$ *of the zeros satisfy RH (Selberg, Levinson, Conrey, Bui, Young).*

Jensen-Pólya Program

J. W. L. Jensen
(1859—1925)

George Pólya
(1887—1985)

Jensen-Pólya Program

Definition

The **Riemann Xi-function** is the entire order 1 function

$$\Xi(z) := \frac{1}{2}\left(-z^2 - \frac{1}{4}\right)\pi^{\frac{iz}{2}-\frac{1}{4}}\Gamma\left(-\frac{iz}{2}+\frac{1}{4}\right)\zeta\left(-iz+\frac{1}{2}\right).$$

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RH is true $\iff$ *all of the zeros of* $\Xi(z)$ *are purely real.*

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Question

Is there a criterion for checking this?

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Definition (Jensen)

If $a : \mathbb{N} \mapsto \mathbb{R}$ is an arithmetic function, then the **Jensen polynomial of degree** d **and shift** n is

$$J_a^{d,n}(X) := \sum_{j=0}^{d} \binom{d}{j} a(n+j) \cdot X^j.$$

Jensen's Criterion

Theorem (Jensen-Pólya (1927))

With $z = -x^2$*, define Taylor coefficients* $\gamma(n)$

$$\Xi_1(x) = \frac{1}{8} \cdot \Xi\left(\frac{i}{2}\sqrt{x}\right) =: \sum_{n \geq 0} \frac{\gamma(n)}{n!} \cdot x^n.$$

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2. *The hyperbolicity is known for $d \leq 3$ by work of Csordas, Norfolk and Varga, and Dimitrov and Lucas.*
3. *Nothing for $d \geq 4$.*

New Theorem

Theorem 1 (Griffin, O, Rolen, Zagier)

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2. *We actually* **"locate"** *the real zeros!*
3. *Wagner is generalizing to general L-functions.*

Hermite Polynomials

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Example

The first few Hermite polynomials

$$\begin{aligned} H_0(X) &= 1, \\ H_1(X) &= 2X, \\ H_2(X) &= 4X^2 - 2, \\ H_3(X) &= 8X^3 - 12X, \\ H_4(X) &= 16X^4 - 48X^2 + 12. \end{aligned}$$

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2. *If $d_1 > d_2$, then there is a zero of $H_{d_1}(X)$ between any two zeros of $H_{d_2}(X)$.*

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2. *If* $d_1 > d_2$, *then there is a zero of* $H_{d_1}(X)$ *between any two zeros of* $H_{d_2}(X)$.
3. *If* S_d *denotes the "suitably normalized" zeros of* $H_d(X)$, *then*

$$S_d \longrightarrow \text{Wigner's Semicircle Law.}$$

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Remark

We define **renormalized** *Jensen polynomials* $\widehat{J}_{\gamma}^{d,n}(X)$.

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Theorem 1 (Griffin, O, Rolen, Zagier)

For each degree $d \geq 1$ *we have that*

$$\lim_{n \to +\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X).$$

For each d, *all but (possibly) finitely many* $J_{\gamma}^{d,n}(X)$ *are hyperbolic.*

Degree 2 Normalized Jensen polynomials

n	$\widehat{J}_{\gamma}^{2,n}(X)$
100	$\approx 3.9586X^2 + 0.6107X - 1.9914$
200	$\approx 3.9772X^2 + 0.4522X - 1.9927$
300	$\approx 3.9841X^2 + 0.3777X - 1.9942$
400	$\approx 3.9877X^2 + 0.3318X - 1.9953$
$\vdots$	$\vdots$
10^8	$\approx 3.9999X^2 + 0.0007X - 2.0000$
$\vdots$	$\vdots$
∞	$H_2(X) = 4X^2 - 2$

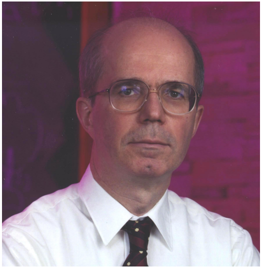
Degree 3 Normalized Jensen polynomials

n	$\widehat{J}_{\gamma}^{3,n}(X)$
100	$\approx 7.8160X^3 + 3.0022X^2 - 11.5732X - 1/2370$
200	$\approx 7.8983X^3 + 2.2409X^2 - 11.7522X - 0.9060$
300	$\approx 7.9288X^3 + 1.8770X^2 - 11.8237X - 0.7526$
400	$\approx 7.9450X^3 + 1.6515X^2 - 11.8625X - 0.6589$
$\vdots$	$\vdots$
10^8	$\approx 7.9999X^3 + 0.0039X^2 - 11.9999X + 0.0015$
$\vdots$	$\vdots$
∞	$H_3(X) = 8X^3 - 12X$

Random Matrix Model Predictions

Freeman Dyson

Hugh Montgomery



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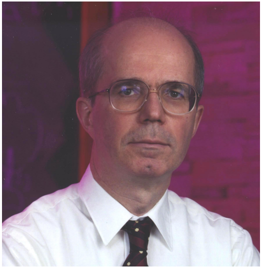

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Gaussian Unitary Ensemble (GUE) (Dyson, Montgomery ('70s))

The nontrivial zeros of $\zeta(s)$ appear to be "distributed like" the eigenvalues of random Hermitian matrices.

Relation to our work

Theorem (Griffin, O, Rolen, Zagier)

GUE is true for the Riemann zeta-function in **derivative aspect**.

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3. *For fixed* d, *we proved that*

$$\lim_{n \to +\infty} \widehat{J}_{\gamma}^{d,n}(X) = H_d(X).$$

4. *The zeros of the* $\{H_d(X)\}$ *and the eigenvalues in GUE both satisfy Wigner's Semicircle Distribution.* □

Computing Jensen Polynomials

Theorem (Pustylnikov (2001), Coffey (2009))

As $n \to +\infty$, we have

$$\xi^{(2n)}(1/2) = \frac{(2n)(2n-1)(2n-2)^{\frac{-1}{4}}}{2^{2n-2}\ln(2n-2)^{\frac{1}{4}}}\left[\ln\left(\frac{2n-2}{\pi}\right) - \ln\ln\left(\frac{2n-2}{\pi}\right) + o(1)\right]^{2n-\frac{3}{2}} \times \exp\left(-\frac{2n-2}{\ln(2n-2)}\right).$$

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Remarks

1. *Derivatives essentially drop to 0 for "small" n.*
2. *After this initial drop, then they have* **exponential growth.**
3. *These asymptotics* **are insufficient** *for approximating $J_\gamma^{d,n}(X)$.*

First 20 Taylor coefficients of $\Xi_1(x)$

m	$\hat{b}_m$
0	6.214 009 727 353 926 (–2)
1	7.178 732 598 482 949 (–4)
2	2.314 725 338 818 463 (–5)
3	1.170 499 895 698 397 (–6)
4	7.859 696 022 958 770 (–8)
5	6.474 442 660 924 152 (–9)
6	6.248 509 280 628 118 (–10)
7	6.857 113 566 031 334 (–11)
8	8.379 562 856 498 463 (–12)
9	1.122 895 900 525 652 (–12)
10	1.630 766 572 462 173 (–13)
11	2.543 075 058 368 090 (–14)
12	4.226 693 865 498 318 (–15)
13	7.441 357 184 567 353 (–16)
14	1.380 660 423 385 153 (–16)
15	2.687 936 596 475 912 (–17)
16	5.470 564 386 990 504 (–18)
17	1.160 183 185 841 992 (–18)
18	2.556 698 594 979 872 (–19)
19	5.840 019 662 344 811 (–20)
20	1.379 672 872 080 269 (–20)

Arbitrary precision asymptotics for $\Xi^{(2n)}(0)$

Notation

1. *Let*

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2. *Following Riemann, we have*

$$\Xi^{(n)}(0) = (-1)^{n/2} \cdot \frac{32\binom{n}{2} F(n-2) - F(n)}{2^{n+2}}$$

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3. *Let* $L = L(n) \approx \log\left(\frac{n}{\log n}\right)$ *be the unique positive solution of the equation* $n = L \cdot (\pi e^L + \frac{3}{4})$.

Arbitrary precision asymptotics

Theorem (Griffin, O, Rolen, Zagier)

To all orders, as $n \to +\infty$, there are $b_k \in \mathbb{Q}(L)$ such that

$$F(n) \sim \sqrt{2\pi}\, \frac{L^{n+1}}{\sqrt{(1+L)n - \frac{3}{4}L^2}}\, e^{L/4 - n/L + 3/4} \left(1 + \frac{b_1}{n} + \frac{b_2}{n^2} + \cdots\right),$$

where $b_1 = \frac{2L^4+9L^3+16L^2+6L+2}{24\,(L+1)^3}$.

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Remarks

1. *The approximation without any b_m improves previous results.*
2. *Using two terms (i.e. b_1) suffices for our RH application.*

Example: $\widehat{\Xi}^{(2n)}(0)$ is the two-term approximation

$2n$	$\widehat{\Xi}^{(2n)}(0)$	$\Xi^{(2n)}(0)$	$\widehat{\Xi}^{(2n)}(0)/\Xi^{(2n)}(0)$
10	$\approx -5.2990317111 \times 10^{-5}$	$\approx -5.3038634278 \times 10^{-5}$	≈ 0.999089019
100	$\approx 4.7698966907 \times 10^{-4}$	$\approx 4.7698430706 \times 10^{-4}$	≈ 1.000011241
1000	$\approx 7.1959898985 \times 10^{236}$	$\approx 7.1959875700 \times 10^{236}$	≈ 1.000000335
10000	$\approx 1.8884738933 \times 10^{4248}$	$\approx 1.8884738827 \times 10^{4248}$	≈ 1.000000005
100000	$\approx 1.6590460773 \times 10^{56328}$	$\approx 1.6590460772 \times 10^{56328}$	≈ 1.000000000

How do these asymptotics imply Theorem 1?

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Theorem 1 is an example of a **general phenomenon**!

Hyperbolic Polynomials in Mathematics

Remark

Hyperbolicity of "generating polynomials" is studied in enumerative combinatorics in connection with **unimodality** *and* **log-concavity**

$$a(n)^2 \geq a(n-1)a(n+1).$$

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- *Group theory (lattice subgroup enumeration)*
- *Graph theory*
- *Symmetric functions*
- *Additive number theory (partitions)*
- ...

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as $n \to +\infty$

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as $n \to +\infty$ for some real numbers $E(n) > 0$ and $\delta(n) \to 0$.

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An $a(n)$ with an asymptotic formula has appropriate growth if

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$$\log\left(\frac{a(n+j)}{a(n)}\right) = A(n)j - B(n)j^2 + o\left(\delta(n)^2\right),$$

where $A(n) > 0$ and $0 < B(n) \to 0$.

General Theorem

Definition

If $a(n)$ has appropriate growth, then the **renormalized Jensen polynomials** are defined by

$$\widehat{J}_a^{d,n}(X) := \frac{2^d}{\delta(n)^d \cdot a(n)} \cdot J_a^{d,n}\left(\frac{\delta(n)X - 1}{E(n)}\right).$$

General Theorem

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$$\widehat{J}_a^{d,n}(X) := \frac{2^d}{\delta(n)^d \cdot a(n)} \cdot J_a^{d,n}\left(\frac{\delta(n)X-1}{E(n)}\right).$$

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Hermite Polynomial Generating Function

Lemma (Generating Function)

We have that

$$e^{2XY-Y^2} =: \sum_{d=0}^{\infty} H_d(X) \cdot \frac{Y^d}{d!}$$

$$= 1 + 2X \cdot Y + (4X^2 - 2) \cdot \frac{Y^2}{2} + (8X^3 - 12X) \cdot \frac{Y^3}{6} + \ldots.$$

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Remark

The rough idea of the proof is to show for large fixed n that

$$\sum_{d=0}^{\infty} \widehat{J}_a^{d,n}(X) \cdot \frac{Y^d}{d!} \approx e^{2XY-Y^2}.$$

Proof of the General Thm

- **Fix** n. The generating function for $J_a^{d,n}(X)$ **for all** d is

$$\mathcal{J}_a(n;X,Y) := \sum_{d\geq 0}\sum_{j=0}^{d} \binom{d}{j} \frac{a(n+j)}{a(n)} \cdot X^j \cdot \frac{Y^d}{d!}.$$

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- Therefore, it suffices to prove

$$\lim_{n \to +\infty} \mathcal{J}_a\left(n; E(n)^{-1}\left(\delta(n)X - 1\right), 2Y\delta(n)^{-1}\right) = e^{2XY - Y^2}.$$

Proof of the General Thm Continued

- We want a "master gen fnc" **for all** $\mathcal{J}_{a,n}(X, Y)$ using

$$\frac{a(n+j)}{a(n)} = E(n)^j \cdot e^{-\delta(n)^2(j^2/4+C(j;n))}.$$

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- Summing in j and letting $C(j) := C(j;n)$ gives "master gen fcn"

$$\mathcal{J}_a(n;X,Y) = \sum_{d\geq 0}\sum_{j=0}^{d} e^{-\frac{\delta^2 j^2}{4}+C(j)} \cdot \frac{(EXY)^j}{j!} \cdot \frac{Y^{d-j}}{(d-j)!}$$

$$= e^Y \sum_{j\geq 0} e^{-\frac{\delta^2 j^2}{4}+C(j)} \cdot \frac{(EXY)^j}{j!}.$$

Proof of the General Thm Continued

- Using the definition of E and δ, one gets

$$\begin{aligned}\mathcal{J}_a(n; E^{-1}(\delta X - 1), 2Y\delta^{-1}) &= \exp(2\delta^{-1}Y)\sum_{j\geq 0} e^{-\frac{\delta^2 j^2}{4}+C(j)} \cdot \frac{(E\cdot E^{-1}(\delta X - 1))^j (2\delta^{-1}Y)^j}{j!}\\ &= \exp(2\delta^{-1}Y)\sum_{j\geq 0} e^{-\frac{\delta^2 j^2}{4}+C(j)} \cdot \frac{(2\delta^{-1}(\delta X - 1)Y)^j}{j!}.\end{aligned}$$

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- Binomial Thm gives

$$\mathcal{J}_a(n; E^{-1}(\delta X - 1), 2Y\delta^{-1}) =$$

$$\exp(2\delta^{-1}Y) \sum_{h,\ell\geq 0} e^{-\frac{(\ell+h)^2\delta^2}{4} + C(\ell+h)} \cdot \frac{(2XY)^h}{h!} \cdot \frac{(-2\delta^{-1}Y)^\ell}{\ell!}.$$

Proof of the General Thm Continued

- In our limit, the relevant part of the red factor is

$$e^{-\frac{(\ell+h)^2\delta^2}{4}+C(\ell+h)} = \sum_{m\geq 0}\left(\frac{-\delta^2}{4}\right)^m \frac{(\ell+h+o(1))^{2m}}{m!}$$
$$= \sum_{m\geq 0}\sum_{0\leq a\leq 2m}\left(\frac{-\delta^2}{4}\right)^m \binom{2m}{a}\frac{\ell^a(h+o(1))^{2m-a}}{m!}.$$

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- Insert this complicated formula for ℓ^a in the red factor

$$\ell^a = \sum_{b=0}^{a}\sum_{c=0}^{b}\binom{\ell}{b}\binom{b}{c}c^a(-1)^{b-c}.$$

Proof of the General Thm Continued

- The red factor becomes

$$e^{-\frac{(\ell+h)^2\delta^2}{4}+C(\ell+h)} = \sum_{\substack{m,a,b,c\geq 0\\ 2m\geq a\geq b\geq c}} \left(\frac{-\delta^2}{4}\right)^m \binom{2m}{a} \frac{(h+o(1))^{2m-a}}{m!} \binom{\ell}{b} \binom{b}{c} c^a (-1)^{b-c}.$$

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$$\begin{aligned}
&\mathcal{J}_a(n; E^{-1}(\delta X-1), 2Y\delta^{-1}) \\
&= \exp(2\delta^{-1}Y) \sum_{\substack{h,m\geq 0\\ 0\leq \ell\leq N\\ a,b,c\geq 0\\ 2m\geq a\geq b\geq c}} \left(\frac{-\delta^2}{4}\right)^m \binom{2m}{a} \frac{(h+o(1))^{2m-a}}{m!c!(b-c)!} c^a(-1)^{b-c} \frac{(2XY)^h}{h!} \frac{(-2\delta^{-1}Y)^\ell}{(\ell-b)!} \\
&= \sum_{\substack{h,m\geq 0\\ a,b,c\geq 0\\ 2m\geq a\geq b\geq c}} \left(\frac{-\delta^2}{4}\right)^m \binom{2m}{a} \frac{(h+o(1))^{2m-a}}{m!c!(b-c)!} c^a(-1)^{b-c} \frac{(2XY)^h}{h!} (-2\delta^{-1}Y)^b + O(Y^N).
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$$= \exp(2XY) \sum_{\substack{m \geq 0 \\ 0 \leq c \leq 2m}} \frac{c^{2m}(-1)^{m-c}}{m!c!(2m-c)!} \cdot Y^{2m} + O(Y^N)$$

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- Miracle #5 is the following complicated formula

$$\sum_{0 \leq c \leq 2m} \frac{c^{2m}(-1)^c}{(2m)!} \binom{2m}{c} = 1.$$

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- Putting this together gives

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- Hyperbolicity follows from facts about Hermite polynomials. □

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Suppose that $a(n)$ has **appropriate growth***.*
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Example

We have that $p(4) = 5$ because the partitions of 4 are

$$4, \quad 3+1, \quad 2+2, \quad 2+1+1, \quad 1+1+1+1.$$

Partition Jensen Polynomials

Example

The roots of the quadratic $J_p^{2,n}(X)$ are

$$\frac{-p(n+1) \pm \sqrt{p(n+1)^2 - p(n)p(n+2)}}{p(n+2)}.$$

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If $n \geq 25$, then $J_p^{2,n}(X)$ is hyperbolic.

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TABLE 1. Conjectured minimal values of $N(d)$

d	1	2	3	4	5	6	7	8	9
$N(d)$	1	25	94	206	381	610	908	1269	1701

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2. *This is a consequence of the* **General Theorem**.

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Example (Partition Generating Function)

We have the weight $-1/2$ modular form
$$f(\tau) = \sum_{n=0}^{\infty} p(n) e^{2\pi i n\tau - \frac{1}{24}}.$$

Jensen polynomials for modular forms

Theorem 3 (Griffin, O, Rolen, Zagier)

Let f be a weakly holomorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with real coefficients and a pole at $i\infty$. Then for each degree $d \geq 1$

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Thm 3 **separates** *these roots using the modified polynomials.*

Asymptotics are known for modular forms

A First Example (Hardy-Ramanujan (1918), Rademacher (1937))

If n is a positive integer, then in terms of a Kloosterman sum $A_k(n)$

$$p(n) = 2\pi(24n-1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right),$$

where $I_{\frac{3}{2}}(\bullet)$ *is the index* $3/2$ *I-Bessel function.*

Our Results

General Theorem (Griffin, O, Rolen, Zagier)

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4. **Any real sequence** *with suitable asymptotics.*