한국수학사학회지 *The Korean Journal for History of Mathematics*
투고 논문

On the summability of infinite series and Hüseyin Bor

Jungoh Lee

In general, there is summability among the mathematical tools that are the criterion for the convergence of infinite series. Many authors have studied on the summability of infinite series, the summability of Fourier series and the summability factors. Especially, Hüseyin Bor had published his important results on these topics from the beginning of 1980 to the end of 1990. In this paper, we investigate the minor academic genealogy of teachers and pupils from Fourier to Hüseyin Bor in section 2. We introduce the Hüseyin Bor's major results of the summability for infinite series from 1983 to 1997 in section 3. In conclusion, we summarize his research characteristics and significance on the summability of infinite series. Also, we present the diagrams of Hüseyin Bor's minor academic genealogy from Fourier to Hüseyin Bor and minor research lineage on the summability of infinite series.

Keywords: Summability of Infinite series, Summability of Fourier series, Summability factor, Absolute summability

1 Introduction

We can see mount Erciyes at Kayseri in Turkey which is covered with snow. The world famous Hüseyin bor was a professor at the Erciyes University named after the mount. He had led extensive research here on the summability factors and the summability method of infinite series and many other things until he retired. Mathematical theory of summability is mainly used in the fields of information and communication and physics. In other words, the theory of the Cesàro summability is used to restore the input signal to the image signal or audio signal. Also, in the theory of control of physics, theory of summability is needed to represent a specific regulator of quantum field which defines

---

MSC: 40D15, 40F05,40G05, 42B15. 42A20, 42A24
제출일 : 2017년 8월 23일

infinite values as finite values. Here are some of the notations needed to discuss the summability of infinite series.

Let the series of a sequence (a_n) be

$$\sum_{n=0}^{\infty} a_n \tag{1}$$

with the sequence (s_n) of partial sums

$$s_n = a_0 + a_1 + \cdots + a_n = \sum_{k=0}^{n} a_k \qquad (k = 0, 1, 2, \cdots).$$

The series is said to be Cesàro means or summable $(C, 1)$ if there exist a limit value of the arithmetic mean for the partial sum s_n such that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} s_k = A.$$

Then n-th Cesàro means of the partial sum s_n is denoted by σ_n such that

$$\sigma_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1}.$$

(1) is said to be summable $|C, 1|_k$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n - \sigma_{n-1}|^k = \sum_{n=1}^{\infty} n^{k-1} |\Delta\sigma_{n-1}|^k < \infty \tag{2}$$

where $k \geq 1$. The n-th Cesàro means of the sequence (u_n) with $u_n = na_n$ can be denoted by t_n such that

$$t_n = n(\sigma_n - \sigma_{n-1}) = n(\Delta\sigma_{n-1}). \tag{3}$$

Thus (2) is can be written as

$$\sum_{n=1}^{\infty} \frac{|t_n|^k}{n} < \infty.$$

Also,

$$\sum_{n=1}^{\infty} n^{\gamma k+k-1} |\Delta\sigma_{n-1}|^k < \infty, \tag{4}$$

can be represented by

$$\sum_{n=1}^{\infty} \frac{|t_n|^{k\gamma+k}}{n} < \infty \tag{5}$$

where $k \geq 1, \gamma \geq 0$. Thus, if one of (4) and (5) is satisfied, (1) is said to be summable $|C, 1, \gamma|_k$. On the other hand, the α order n-th Cesàro mean of the sequence (s_n) is denoted by σ_n^α. Let $\alpha > -1, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty$$

then the series (1) is said to be summable $|C, \alpha|_k$. Let (p_n) be a sequence of positive real constants such that

$$P_n = p_0 + p_1 + \cdots + p_n = \sum_{\nu=0}^{n} p_\nu \to \infty, \quad n \to \infty. \tag{6}$$

Through the sequence-to-sequence transformation, which is used to improve the speed of convergence,

$$T_n = {P_n}^{-1} \sum_{\nu=0}^{n} p_\nu s_\nu \quad (P_n \neq 0) \tag{7}$$

defines the sequence (T_n) of the $(\bar{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficient (p_n). Then, the series (1) is said to be summable $|\bar{N}, p_n|_k$, where $k \geq 1$ if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta T_{n-1}|^k < \infty. \tag{8}$$

And if

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\gamma k+k-1} |\Delta T_{n-1}|^k < \infty \tag{9}$$

then the series (1) is said to be summable $|\bar{N}, p_n; \gamma|_k$ where $\gamma \geq 0$, $\Delta T_{n-1} = T_n - T_{n-1}$. From the Riesz mean (7) of the sequence (s_n), if the sequence (T_n) such that

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta T_{n-1}|^k < \infty \tag{10}$$

then the series (1) is said to be summable $|R, p_n|_k$. The series (1) is said to be summable (N, p_n) if T_n of (7) is

$$\lim_{n \to \infty} T_n = S.$$

Also, the series (1) is said to be bounded $[\bar{N}, p_n]_k$ if

$$\sum_{\nu=1}^{n} p_n |s_\nu| = O(P_n), \quad n \to \infty. \tag{11}$$

For a positive integer n, a sequence (b_n) is said to convex if

$$\Delta b_n = b_n - b_{n+1}, \qquad \Delta^2 b_n \geq 0, \quad (\Delta^2 b_n = \Delta b_n - \Delta b_{n+1}) \tag{12}$$

A given sequence (b_n) of positive numbers is said to be quasi monotone if

$$\left(\frac{b_n}{n^\alpha}\right)$$

is not increasing where $\alpha \geq 0$.

Also, the sequence (b_n) is said to be δ-quasi monotone for a positive sequence (δ_n) if

$$b_n \to 0, \quad 0 < b_n, \quad -\delta_n \leq \Delta b_n. \tag{13}$$

2 The minor academic genealogy from Fourier to Hüseyin Bor

In this section, we investigate the teacher-student relations that lasted for about 170 years from Fourier in 1807 to Hüseyin Bor in 1982. First, Jean-Baptiste Joseph Fourier was taught by Lagrange, Laplace and Monge at the École Normale Supérieure in France. He studied on the conduction of heat in solids in 1807. Fourier is most influenced by Lagrange. There are Fourier's juniors about 56,978.[1)] There were two Lagrange's pupils. One of them is Dirichlet. He was taught by Fourier and Poisson in France. He studied on the partial results of index 5 on the final theorem of Pierre Fermat.[2)] He returned to Germany and earned a degree in 1827 from the university of Rheinische Friedrich-Wilhelms in Bonn. Among Dirichlet's disciples, Lipschitz studied the determination of the state of magnetic forces moving into ellipsoids[3)] and earned a degree from the university of Berlin in 1853. Meanwhile, Klein was the only pupil of Lipschitz. He earned his degree at the university in 1868 by studying the conversion of quadratic general equations between line coordinates and standard forms.[4)] He had his students about 63. One of them is Lindemann. Lindemann studied infinitely small motion and force systems using general projection measurements[5)] and obtained a degree at the university of Nürnberg in 1873. Oskar Perron, one of Lindemann's pupils, earned a degree from München university by studying on the rotation of a steel at the center of gravity by external forces[6)] in 1902. His student, Berki Yurtsever, earned a degree in 1941 from the

1) Mathematics Genealogy Project, Department of Mathematics North Dakota State university.

2) Partial Results on Fermat's Last Theorem, Exponent 5.

3) Determinatio status magnetici viribus inducentibus commoti in ellipsoide.

4) Über die Transformation der allgemeinen Gleichung des zweiten Grades zwischen Linien Koordinaten auf eine kanonische Form.

5) Über unendlich kleine Bewegungen und über Kraftsysteme bei allgemeiner projektivischer Maßbestimmung.

6) Über die Drehung eines starren Körpers um seinen Schwerpunkt bei Wirkung äußerer Kräfte.

university of München by studying solution of partial differential equations by infinite series.[7] Berki Yurtsever returned to Ankara university in Turkey where he taught his only disciple Hüseyin Bor. Hüseyin Bor earned a degree from Ankara university in 1982 by studying absolute summability methods and summability factors. Since then, he has been a faculty at the university of Erciyees and world-renowned for his outstanding work on the 'the summability of infinite series', 'the summability of Fourier series', 'the summable factors'.

3 Hüseyin Bor's research from 1983 to 1997

This section focuses on Hüseyin Bor's research for about 15 years since 1983. After earning his degree in 1982, he had published 15 research papers on the summability of infinite series while he was studying at the university of Birminghman in England until 1985. First, he introduced on 'the absolute summability factors of infinite series' [17] and 'a note on $|\bar{N}, p_n|_k$ summability factors for infinite series' [1] in 1983. He also introduced the concept of absolute $|\bar{N}, p_n|_k$ summability of order $k, k \geq 1$ and generalized summability. He studied on $|\bar{N}, p_n|_k$ summability factors[12] in 1985. This result is a more generalization of the results of T. Singh.[8] If $k = 1$, then T. Singh's theorem and his theorem of [12] are the same. In [12], He showed the result by using $P_\mu |b_\mu| = O(1)$ as $\mu \to \infty$. If the series (1) is bounded $[\bar{N}, p_n]_k, k \geq 1$ as (11) and the sequences (b_n) and (p_n) of (6) are satisfied the following two conditions:

$$\sum_{n=1}^{\mu} p_n |b_n| = O(1),$$

$$P_\mu |\Delta b_\mu| = O(p_\mu |b_\mu|),$$

then the series $\sum_{n=1}^{\infty} a_n P_n b_n$ is summable $|\bar{N}, p_n|_k$ as (8). He has remarked that $|\bar{N}, p_n|_k$ summability is more general than those of absolute Cesàro summability $|C, 1|_k$ of order one and absolute Riesz summability $|R, p_n|$ of order one. In 1985, Hüseyin Bor[16] also proved that summable $|\bar{N}, p_n|_k, k > 1$ by the following assumptions. Let (p_n) be a sequence of positive real constants such that

$$np_n = O(P_n), \tag{14}$$

7) Lösung einer partiellen Differentialgleichung durch unendliche Reihen.
8) A note on $|\bar{N}, p_n|$ summability factors for infinite series, 3. Math. Soc. 12-13 (1977-78), 25-28.

$$P_n = O(np_n) \tag{15}$$

where $n \to \infty$. If (1) is summable $|C,1|_k$ then it is also summable $|\bar{N},p_n|_k, k > 1$. In general, if $p_n \neq 1, n \in N$ for at least one, it is known that the summability $|\bar{N},p_n|_k$ and the summability $|C,1|_k$ are independent of each other. In 1986([2]), he proved that the series (1) is summable $|\bar{N},p_n|_k$ then the series is also summable $|C,1|_k$. This result is the converse of [16]. He generalized a theorem of Ming-Po Chen[9)] on $|\bar{N},p_n|$ summability factors of infinite series under weaker conditions in [15]. Chen introduced the following his result. Let the series (1) be bounded $[\bar{N},p_n]$ and (p_n) be a positive non-increasing sequence such that $P_n \to \infty$. If (b_n) is a convex of (12) such that $\sum_{n=1}^{\infty} p_n b_n < \infty$, $\frac{1}{n} = O(p_n)$ and $\Delta\left(\frac{1}{p_n}\right) = O(1), \quad n \to \infty$ then the series

$$\sum_{n=0}^{\infty} a_n b_n \tag{16}$$

is also summable $|\bar{N},p_n|$. On the other hand, Hüseyin Bor introduced the following result in [15]. Let the series (1) be bounded $[\bar{N},p_n]$ and (p_n) be a positive sequence such that $P_n \to \infty$. Let (b_n) and (c_n) be sequences such that $|\Delta b_n| \leq c_n$, $c_n \to 0$ as $n \to \infty$, $\sum_{n=1}^{\infty} nP_n|\Delta c_n| < \infty$ and $P_n|b_n| = O(1)$. If (p_n) satisfies $\frac{1}{n} = O(p_n)$ then the series (16) is summable $|\bar{N},p_n|$. He introduced the sequence (c_n) so that $|\Delta c_n|$ is much smaller then $|\ \Delta^2 b_n|$. Eventually, the relation of the following two conditions

$$\sum_{n=1}^{\infty} nP_n|\Delta c_n| < \infty, \tag{17}$$

$$\sum_{n=1}^{\infty} nP_n|\Delta^2 b_n| < \infty \tag{18}$$

is (17) weaker then (18). Meanwhile, since the end of 1980, Hüseyin Bor extends his research on the possibility of summation. In 1989, he studied on summability of Fourier series([5],[6]). He also published eleven his research papers in 1991. These papers mainly present excellent results that have generalized the results of other researchers. In 1991, he studied on factors for $|\bar{N},p_n|_k$ summability of infinite series. W.T. Sulaiman has proved summability factor of infinite series for $|C,1| = |C,1|_1$. Hüseyin Bor generalized

9) On $|\bar{N},p_n|$ summability factors of infinite series, *Math. Res. Center, National Taiwan univ. Hung-Ching Chow*, 65(1967), 114-120.

Sulaiman's result[10)] on summability factors in [4]. He made the following assumption

$$\sum_{n=0}^{\infty}\left(\frac{p_n}{P_n}\right)(|\alpha_n|d_n)^k < \infty$$

instead of

$$\sum_{n=1}^{\infty}\left(\frac{d_n|\alpha_n|}{n}\right) < \infty$$

where $(d_n)_{n\geq 0}$ is a given sequence of positive numbers and (α_n) is a sequence of complex numbers. In the same year, he generalized the results of L.S. Bosanquet[11)] in [14]. Bosanquet proved in order that $|R, p_n|_k$ of (10) then $|R, q_n|_k$ if and only if

$$\frac{q_n P_n}{p_n Q_n} = O(1)$$

in case of $k = 1$,

$$Q_n = q_0 + q_1 + \cdots + q_n = \sum_{\nu=0}^{n} q_\nu \to \infty, \quad n \to \infty. \tag{19}$$

Hüseyin Bor[14] also extended $|\bar{N}, p_n|_k$ summability to $|\bar{N}, p_n, \gamma|_k$ summability by introducing $\gamma \geq 0$. Of course, a summability method A is said to be weaker than another summability method B, if the summability of a series by the method A implies its summability by the method B.

On the other hand, S.M. Mazhar[12)] had studied for $|C, 1|_k$ summability factors of infinite series. In 1993, Hüseyin Bor[9] generalized the theorem of Mazhar. He has proved that the series $\sum a_n\alpha_n$ is summable $|\bar{N}, p_n|_k$, $k \geq 1$ by using the condition

$$\sum_{n=1}^{m}\frac{p_n}{P_n}|t_n|^k = O(\log m), \quad m \to \infty$$

instead of

$$\sum_{n=1}^{m}\frac{1}{n}|t_n|^k = O(\log m), \quad m \to \infty.$$

Also, he generalized the result of δ-quasi monoton[13)] of (13) on $|\bar{N}, p_n; \gamma|_k$ summability factors of infinite series in [8]. In this result, he assumed that the sequence (p_n)

10) Multipliers for $|C, 1|$ summability of Jacobi series, *Indian J. Pure Äppl. Math.*, 18(1987), 1121-1130.
11) *MR 11* (1950), 654.
12) On $|C, 1|_k$ summability factors ofinfinite series, *Indian J. Math.* 14 (1972), 45-48.
13) On quasi-monotone sequences and their applications, *Bull. Aust. Math. Soc.* 43(1991) 187–192.

satisfies the following two conditions

$$P_n = O(np_n), \quad n \to \infty$$

and

$$\sum_{n=\nu}^{\infty} \left(\frac{P_n}{p_n}\right)^{\gamma k-1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_\nu}{p_\nu}\right)^{\gamma k} \frac{1}{P_\nu}\right). \tag{20}$$

He has proved that the series

$$\sum_{n=0}^{\infty} a_n \lambda_n$$

is summable $|\bar{N}, p_n; \gamma|_k$ of (9) where $k \geq 1$, $\gamma \geq 0$ if

$$\sum_{n=1}^{\mu} \left(\frac{P_n}{p_n}\right)^{\gamma k-1} |t_n|^k = O(X_m) \quad \mu \to \infty$$

for a positive increasing sequence (X_m), $\quad n \to \infty$.

In the following year, he obtained the results of a collaboration with M.A. Sarigol in [3] at Pamukkale university. They have proved that

$$\lambda \in (|\bar{N}, p_n|, |\bar{N}, q_n|_k)$$

is equivalent to

$$(i) \quad \lambda_n = O(1)$$
$$(ii) \quad \Delta\lambda_n = O\left(\frac{p_n}{P_n}\right)$$
$$(iii) \quad \lambda_n = O\left(\left(\frac{p_n}{P_n}\right)\left(\frac{Q_n}{q_n}\right)^{\frac{1}{k}}\right)$$

and

$$(iv) \quad \sum_{\nu=1}^{\infty} \left(\frac{p_\nu}{P_\nu}\right) \left|\frac{P_\nu}{p_\nu}\Delta\lambda_\nu + \lambda_{\nu+1}\right|^{k^*} < \infty$$
$$(v) \quad \sum_{\nu=1}^{\infty} \left(\frac{p_\nu}{P_\nu}\right) \left\{\frac{q_\nu P_\nu}{p_\nu Q_\nu}|\lambda_\nu|\right\}^{k^*} < \infty$$

in cases of $1 \leq k < \infty$ and $1 < k < \infty$ respectively where Q satisfied (19) $k^* = \frac{k}{k-1}$ is the conjugate index of k .

Also, he introduced on $|\bar{N}, p_n; \delta|_k$ summability factors as the general theorem of the result of $|\bar{N}, p_n|_k$ summability factor([10]) in [11]. He supposed that (X_n) is a positive non decreasing sequence and there exists sequences (λ_n) and (β_n) such that

$$|\Delta\lambda_n| \leq \beta_n, \quad \beta_n \to 0, \quad n \to \infty$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n < \infty$$

$$|\lambda_n|X_n = O(1), \quad n \to \infty.$$

If (p_n) is a sequence is satsfied

$$P_n = O(np_n), \quad n \to \infty$$

and

$$\sum_{n=\nu+1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\gamma k-1} \frac{1}{P_{n-1}} = O\left(\left(\frac{P_\nu}{p_\nu}\right)^{\gamma k} \frac{1}{P_\nu}\right),$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = O(V_m) \quad m \to \infty,$$

then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n; \delta|_k$ for $0 \leq \delta < \frac{1}{k}$ and $k \geq 1$. Where

$$t_n = \frac{1}{n+1}\sum_{\nu=1}^{n} \nu a_\nu.$$

$|\bar{N}, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n; \gamma|_k$ summability when $\delta = 0$. In addition to, $|\bar{N}, p_n; \delta|_k$ summability is the same as $|C, 1; \delta|_k$ summability when $p_n = 1$.

Subsequently, he published his research on Fourier series.([13],[7])

4 Conclusion: The Characteristics of Hüseyin Bor's Study on the summability of Infinite Series

The summability study is more than 200 years old. The study of summability had been introduced in various ways by Leonhard Euler, Niels Henrik Abel, Georg Friedrich Bernhard Riemann and Niels Erik Nörlund. Also, Ernesto Cesàro, Godfrey Harold Hardy introduced the Cesàro summability by defining partial sum of infinite series as the limit of the arithmetic mean in 1890. Also, Lipot Fejér,[14)] Henri Léon Lebesgue,[15)] and Émile Borel,[16)] also studied the summability.

On the other hand, Hüseyin Bor published about 90 papers between 1983 and 1997

14) Sur les functions bornée set intêgrables, *C.R. Acad. Sci. Paris*, 131(1900), 984-987.
15) Researches sur la convergence des series de Fourier, *Math. Annalen*, 6(1905), 251-280.
16) Lecons suries series divergentes, 2nd Edition, Paris, 1928.

through his research on the summability methods and the summability factors. There are two theorems ([2] and [16]) that stand out for Hüseyin Bor's work. He had proved the following result. Let $n \to \infty$ and the sequence (p_n) in (6) be satisfied with (14) and (15). If the series (1) is $|C,1|_k$ summable then the series (1) is $|\bar{N},p_n|_k$ summable. And the inverse of this result is the following. If the series (1) is $|\bar{N},p_n|_k$ is summable, then (1) is $|C,1|_k$ summable, where $k \geq 1$. And these two results eventually expand as follows. Let sequence (p_n) be satisfied with the expressions (14), (15) and (20). If the series (1) is $|C,1;\gamma|_k$ summable, also (1) is $|\bar{N},p_n;\delta|_k$ summable, where $\delta \geq 0$ and $k \geq 1$. And Let the sequence (p_n) be satisfied with (14) and

$$\sum_{\nu=1}^{n} \left(\frac{P_\nu}{p_\nu}\right) = O(P_{n-1}) \tag{21}$$

which is weaker than (17). Here, if the sequence (p_n) is satisfied with (15), then the sequence satisfies (21) as well. But the reverse is not true. Thus, let the sequence (p_n) be satisfied with (14). The following equations needed to prove these resuls. The equation derived from (7) for the $(\bar{N},p_n)$ mean of the sequence (s_n) is as follows.

$$\begin{aligned} T_n &= \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu s_\nu \\ &= \frac{1}{P_n}\sum_{\nu=0}^{n} p_\nu \sum_{\mu=0}^{\nu} a_\mu \\ &= \frac{1}{P_n}\sum_{\nu=0}^{n}(P_n - P_{\nu-1})a_\nu, \quad n \geq 0. \end{aligned}$$

for $n \geq 1$, ΔT_{n-1} is

$$\begin{aligned} \Delta T_{n-1} &= T_n - T_{n-1} \\ &= \frac{1}{P_n}\sum_{\nu=1}^{n} p_\nu s_\nu - \frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1} p_\nu s_\nu \\ &= \frac{1}{P_n}\sum_{\nu=1}^{n}(P_n - P_{\nu-1})a_\nu - \frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1}(P_{n-1} - P_{\nu-1})a_\nu \\ &= \frac{1}{P_n}\sum_{\nu=1}^{n} P_n a_\nu - \frac{1}{P_n}\sum_{\nu=1}^{n} P_{\nu-1}a_\nu - \frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1} P_{n-1}a_\nu + \frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1} P_{\nu-1}a_\nu \\ &= \sum_{\nu=1}^{n} a_\nu - \frac{1}{P_n}\sum_{\nu=1}^{n} P_{\nu-1}a_\nu - \sum_{\nu=1}^{n-1} a_\nu + \frac{1}{P_{n-1}}\sum_{\nu=1}^{n-1} P_{\nu-1}a_\nu \end{aligned}$$

$$= \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1} a_\nu - \frac{1}{P_n} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu + \left(\sum_{\nu=1}^{n} a_\nu - \sum_{\nu=1}^{n-1} a_\nu \right)$$
$$= \frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1} a_\nu - \frac{1}{P_n} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu + a_n$$
$$= \left(\frac{1}{P_{n-1}} \sum_{\nu=1}^{n-1} P_{\nu-1} a_\nu + a_n \right) - \frac{1}{P_n} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu$$
$$= \frac{1}{P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu - \frac{1}{P_n} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu$$
$$= \left(\frac{1}{P_{n-1}} - \frac{1}{P_n} \right) \sum_{\nu=1}^{n} P_{\nu-1} a_\nu$$
$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{n} P_{\nu-1} a_\nu, \quad 1 \le n$$

As the same way, K_n is the $(\bar{N}, q_n)$ mean of the sequence (s_n) such that

$$K_n = {Q_n}^{-1} \sum_{\nu=1}^{n} Q_\nu s_\nu = \frac{1}{Q_n} \sum_{\nu=1}^{n} (Q_n - Q_{\nu-1}) a_\nu$$

and

$$\Delta K_{n-1} = K_n - K_{n-1} = \frac{q_n}{Q_n Q_{n-1}} \sum_{\nu=1}^{n} Q_{\nu-1} a_\nu .$$

In conclusion, Hüseyin Bor's research has three major achievements on the summability of infinite series.

First, Hüseyin Bor was the first who begun the study of $|\bar{N}, p_n|_k$ summability and obtained a relation between $|\bar{N}, p_n|_k$ and $|C, 1|_k$ summability. The conception of the absolute $(\bar{N}, p_n)$ summable is expressed by $|\bar{N}, p_n|_k$ summable where the order is k and $k \geq 1$.

Second, he established the summability of $|C, 1|_k$ and $|\bar{N}, p_n|_k$. In the order of $k = 1$, he introduced a more general theorem than the absolute cesaro summability of $|C, 1|_k$ and summability of $|R, p_n|_k$. In special cases, when $p_n = 1$ for all n the summability of $|\bar{N}, p_n|_k$ is the summability of $|C, 1|_k$.

Third, he introduced $\gamma \geq 0$ and expanded the summability of $|\bar{N}, p_n|_k$ to $|\bar{N}, p_n, \gamma|_k$. Especially when $\gamma = 0$, the summability of $|\bar{N}, p_n, \gamma|_k$ is the summability of $|\bar{N}, p_n|_k$.

fig 1: The minor academic genealogy from Fourier to Hüseyin Bor(1807-1982)

fig 2:The minor research lineage of Hüseyin Bor on the summability of infinite series(1983-1997)

참고 문헌

1. Hüseyin Bor, A note on $|\bar{N}, p_n|_k$ summability factors for infinite series, *Jour. Math. Sci.*, 16(1983), 70-73.
2. Hüseyin Bor, A note on two summability methods, *Proc. Amer. Math. Soc.*, 98(1)(1986), 81-84.
3. Hüseyin Bor and M.A Sarigol, Characterization of absolute summability factors, *J. Math. Anal.Appl.*, 195(1995), 537-545.
4. Hüseyin Bor, Factors for $|\bar{N}, p_n|_k$ summability of infinite series, *Annales Academiae Scientiarum Fennicae Series A. I. Mathematica Volumen*, 16(1991), 151-154.
5. Hüseyin Bor, Local property of $|\bar{N}, p_n|_k$ summability of factored Fourier series, *Bull. Inst. Math. Acad. Sinica*, 17(1989), 165-170.
6. Hüseyin Bor, Multipliers for $|\bar{N}, p_n|_k$ summability of Fourier series, *Bull. Inst. Math. Acad.Sinica*, 17(1989), 285-290.
7. Hüseyin Bor, On absolute Cesàro summability factors of infinite series and Fourier series, *Bull.Inst.Math. Acad. Sinica*, 26(1998), 195-203.
8. Hüseyin Bor, On absolute summability factors of infinite series, *Proc. Indian Acad. Sci. (Math. Sci.)*, 104(2)(1994), 367-372.
9. Hüseyin Bor, On absolute summability factors, *Proc. Amer. Math. Soc.*, 118(1)(1993), 71-75.
10. Hüseyin Bor, On $|\bar{N}, p_n|_k$ summability factors, *Kuwait J. Sci. Engng.* , 23(1996), 1-5.
11. Hüseyin Bor, On $|\bar{N}, p_n; \delta|_k$ summability factors of infinite series, *Taiwanese J. Math.*, 1(3)(1997), 327-332.
12. Hüseyin Bor, On $|\bar{N}, p_n|_k$ summability factors, *Proc. Amer. Math. Soc.*, 94(3)(1985), 419-422.
13. Hüseyin Bor, On the local property of factored Fourier series, *Z. Anal. Anwendungen*, 16(1997), 769-773.
14. Hüseyin Bor, On the relative strength of two absolute summability methods, *Proc. Amer. Math. Soc.*, 113(4)(1991), 1009-1012.
15. Hüseyin Bor, On the $|\bar{N}, p_n|$ summability factors of infinite series, *Proc. Indian Acad. Sci.(Math.Sci.)*, 98(1)(1988), 53-57.
16. Hüseyin Bor, On two summability methods, *Math. Proc. Cambridge Philos. Soc.*, 97(1985), 147-149.
17. Hüseyin Bor, The absolute summability factors of infinite series, *Pure Appl. Math. Sci.*, 18(1983), 75-79.
18. https://www.genealogy.math.ndsu.nodak.edu.

Department of liberal arts, Chosun College of Science and Technology
E-mail: