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Reformulation of General Relativity in Accordance with Mach's Principle

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It is argued that Einstein's Theory of General Relativity as it stands incorporates Mach's Principle. The boundary conditions for Machian solutions are stated in a coordinate system in which the cosmological background is described by a conformally flat metric. The metric tensor $g_{\mu\nu}$ is then written as a product of the scalar density φ^2 and a tensor density $\gamma_{\mu\nu}$ with unit determinant. In the coordinate system that has been so chosen φ describes the cosmological structure, while $\gamma_{\mu\nu}$ refers to gravitational phenomena. This becomes clear when Einstein's fundamental equations are rewritten in terms of φ and $\gamma_{\mu\nu}$. Then $\kappa\varphi^{-1}$ is seen to play the role of the gravitational constant instead of κ in the weak field approximation. The quantity $\kappa\varphi^{-1}$ can be expressed in terms of the radius and the total mass of the universe and the sign of the forces between inhomogeneities of the metric is determined by the requirements of Mach's principle. The forces which derive from φ are found to be repulsive for the cosmological background, leading to the expansion of the universe, while attractive gravitational forces arise from the deviations of $\gamma_{\mu\nu}$ from the Minkowski metric. Various statements associated with Mach's Principle are discussed in the light of this reformulation of Einstein's Theory.

I. INTRODUCTION

The Mach-Einstein doctrine, which has come to be known as Mach's principle, holds that the basic inertial frame is defined by distant bodies (Mach's fixed stars in the original formulation, now to be understood as the galaxies). According to this view, all inertial effects arise as a consequence of accelerations relative to the system of distant galaxies. In particular, inertia of matter is due entirely to the mutual action of matter. It has been suggested by Einstein that such a mutual action arises from gravitational forces. Then, inertial forces on a body would reduce to gravitational forces exerted by galaxies when the body and the galaxies are in relative accelerated motion. Since, according to this point of view, the need for distinguishing between inertial and gravitational forces disappears, the weaker principle of equivalence follows from the stronger Mach-Einstein principle.

The extent to which Einstein's theory of gravitation, based on the principle of equivalence, incorporates Mach's principle in its strong form is, to this day, not well known, although it has been extensively discussed by many authors, notably by Einstein (*1, 2*), Thirring and Lense (*3*), and more recently by Hönl (*4*), Pirani (*5*), Bondi (*6*), Tangherlini (*7*), Brans (*8*), and Wheeler (*9*) among others. The weak field solutions do reflect Mach's principle in a weak form, showing a basic similarity between inertial forces and gravitational forces generated by the acceleration of distant bodies, without however leading to the complete identification of inertial with gravitational effects. The weak field theory further allows solutions for gravitational fields of massive bodies with arbitrary values of the mass in an otherwise empty space. From the point of view of the Mach-Einstein doctrine these are objectionable features of the solutions. On the hand, when other masses are piled in the neighborhood of a test body, Einstein (*2*) has shown that the inertial mass of the latter increases, so that inertia must be due, at least in part, to the existence of other massive bodies in the universe. The work of Thirring and Lense (*3*) has also demonstrated that inertial forces due to rotation with respect to an inertial frame partly originate in the gravitational forces exerted on the test body by rotating masses in the universe. These examples show that, as far as Mach's principle is concerned, Einstein's theory offers possibilities that do not exist in the Newtonian theory.

Lately, various authors have attempted to construct new theories of gravitation designed to incorporate explicitly Mach's principle in its strong form. In this category we may cite Sciama's (*10, 11*) ingenious vector theory, which is more of an illustrative model than a theory of gravitation, since it leads to repulsive forces. More realistic theories have been proposed by Hoyle (*12, 13*) and Brans and Dicke (*14*). All such theories introduce nongeometrical entities such as new scalar or vector fields superimposed on the metric tensor, thereby destroying the direct relation between the curvature and the distribution of matter that is an essential feature of Einstein's purely geometrical theory of gravitation. Furthermore, a certain degree of arbitrariness inevitably accompanies the introduction of such new fields since the form and the strength of their coupling to the metric tensor are not determined by the principles of General Relativity.

Our aim in this paper is to investigate further Einstein's Theory in the light of Mach's principle without restricting ourselves to the weak field approximation. We propose to exhibit new solutions satisfying Mach's principle in its strong form. These Machian solutions obey certain boundary conditions at spatial infinity, the precise definition of which is one of our main tasks.

In order to find Machian solutions, we proceed in three steps. Firstly, since our purpose is to examine Mach's principle within the framework of General Relativity, we have to find a way of separating local effects from the general cosmological structure due to the distribution of distant bodies, because all statements related to Mach's principle involve such a separation. Secondly, the boundary conditions being only meaningful in a definite coordinate system, we must be able to introduce privileged coordinate frames determined by the over-all cosmological structure that has been separated in the first step. These are the inertial frames that, according to Mach, are determined, to within a kinematical group, by the over-all distribution of matter. Thirdly, to preserve the general covariance of the theory, we have to show that Machian boundary conditions can also be generalized to an arbitrary coordinate system, that is, to noninertial frames.

The key to the success of this program lies in the observed simplicity of the universe at large. A separation of the cosmological background from the local irregularities of the geometry of space-time is made possible by the remarkable uniformity in the distribution of galaxies, an observational fact expressed by the cosmological principle. Roughly, the metric can then be regarded as having a part $C_{\mu\nu}$ which describes a geometry which is conformally flat and spatially homogeneous, and another part referring to deviations from this uniform structure. The inertial frame can then be defined as one in which $C_{\mu\nu}$ takes a conformal form, so that light in this system travels on a straight line with velocity c. The boundary conditions for the metric will now require the $g_{\mu\nu}$ to tend asymptotically to a conformal metric characteristic of a uniform cosmological structure in the inertial frame. Finally, in a general coordinate system, $g_{\mu\nu}$ should tend to $C_{\mu\nu}$ which describe the cosmological background in a noninertial frame. This last point brings us to a re-examination of the meaning of general coordinate transformations in General Relativity. Essentially the point we want to make is that, an acceleration, namely, a transformation that takes the observer from an inertial to a noninertial frame, should be interpreted as a transformation which distorts the uniform and isotropic aspect of the cosmological background. The redistribution of cosmological matter implied by such a transformation will then result in additional gravitational effects which manifest themselves as inertial forces. Einstein's statement that physical laws should be valid in any noninertial system will be read, from a Machian standpoint as: "The physical laws should continue to hold in any cosmological background". The foregoing discussion forms the basis of Section II.

Now comes a crucial observation. Because the cosmological structure is conformally flat, we are led to the following boundary condition at spatial infinity in the inertial system,

$$g_{\mu\nu} \rightarrow \lambda^2 \eta_{\mu\nu} \tag{1.1}$$

where the function λ belongs to the cosmological line element. It also follows that

$$\varphi \rightarrow \lambda \tag{1.2}$$

where

$$\varphi = (-g)^{1/8} = (-\mathrm{Det} \| g_{\mu\nu} \|)^{1/8}, \tag{1.3}$$

g being as usual the determinant of the metric tensor. Thus we must also have

$$\gamma_{\mu\nu} \rightarrow \eta_{\mu\nu} \tag{1.4}$$

with the definition

$$\gamma_{\mu\nu} = (-g)^{-1/4} g_{\mu\nu}, \quad (\text{Det} \, \| \gamma_{\mu\nu} \| = -1) \tag{1.5}$$

Therefore, as long as we work in the inertial system all the information about the cosmological structure is contained in the determinant of the metric and the quantities $\gamma_{\mu\nu}$ describe local irregularities of structure. This suggests that, to give mathematical expression to the separation between local and global effects we can begin by re-expressing Einstein's field equations and the geodesic equation of motion for a test particle in terms of the quantities φ and $\gamma_{\mu\nu}$. We note that φ is not a true scalar and $\gamma_{\mu\nu}$ not a true tensor but rather scalar and tensor densities, respectively, with appropriate weights. It turns out that the same fields also permit a linearization of the field equations even in the absence of the weak field approximation. Hence it is important that we reformulate General Relativity in terms of these quantities. This is done in Section III. It also appears that the function φ shows up as an inertial coefficient in the equation of motion. This is directly relevant to Mach's principle as φ, not being a true scalar, will change when masses in the universe are redistributed by a coordinate transformation. Thus, the inertia of a test particle will depend on the cosmological structure. It is shown later in the paper that in virtue of our boundary conditions, the inertial mass of a particle in an otherwise empty universe vanishes.

Section IV is devoted to the study of the properties of a special cosmological background uniform in space and time in agreement with the Perfect Cosmological Principle. This is known to be a de Sitter Universe. It is a special conformally flat universe with metric of the form (1.1) with $\lambda = \Phi$, Φ being a definite function of the Lorentz invariant length. It is shown that, although the de Sitter universe cannot contain stable matter, it may be interpreted as being associate with a uniform distribution of mass scintillations, that is, unstable masses that give use to a $\delta^{(4)}(x)$ singularity in the equation determining the metric. In the case of a spatially closed de Sitter world a total mass may then be defined. The metric is expressed in terms of the radius of curvature and the total mass in that case, all the mass coming from mass scintillations.

In Section V we turn to the discussion of a conformally flat cosmological background which satisfies the more restricted form of the cosmological principle. In the same section the possibility of a scalar theory of gravitation compatible with Mach's principle is investigated. This question has already been considered by various authors (*15*, *16*, *17*), without, however, the restriction imposed by Mach's principle. The problem is reduced to that of formulating a gravitational theory in conformally flat space-time, since in this case, going to the inertial system, we dispose only of one function

$$\xi = \lambda - \varphi \tag{1.6}$$

to describe the deviations of the metric from the spatially homogeneous structure characterized by λ. If ξ is the field due to an inhomogeneity of the geometry (such as a massive body embedded in an otherwise homogeneous universe), Mach's boundary condition (1.2) implies that $\xi \to 0$ at spatial infinity. We show by studying the equation of motion that a test particle is *repelled* by the massive body whatever the sign of the original constant in Einstein's field equations. This shows that there cannot be attractive forces between massive bodies in a conformally flat space-time. Turning the argument around, we conclude that repulsion between galaxies strengthens our original model of a universe which is roughly conformally flat and in which Mach's principle is valid.

In Section VI we turn to the actual space-time structure where $\gamma_{\mu\nu} \neq \eta_{\mu\nu}$, so that we are no longer in the conformal case. It is shown that the existence of the tensor field

$$h_{\mu\nu} = \gamma_{\mu\nu} - \eta_{\mu\nu} \tag{1.7}$$

leads to attraction between a test particle and massive body producing the tensor field. Thus, a tensor theory of gravitation is necessary to describe local gravitational phenomena in space-time. It is also shown that the solution for the field of a massive body (like the sun) in presence of a homogeneous universe is in agreement with Schwarzschild's solution in isotropic coordinates. Further it is shown that the effective gravitational constant is proportional to the ratio of the effective radius of the universe to its total mass as in Sciama's model (*9*) and in the theories proposed by Jordan (*18*), Dicke (*19*, *20*), and Brans and Dicke (*13*). In our nonstatic model, the gravitational constant is also found to be time dependent as anticipated by Milne (*21*) and Dirac (*22*).

Section VII illustrates how, according to Mach and Einstein, inertial forces due to acceleration (a uniform acceleration in our example) may be interpreted as the gravitational force exerted by a cosmological background which appears anisotropic and accelerated with the opposite acceleration.

Finally, in Section VIII we make a brief comparison between this theory, entirely based on General Relativity, and other theories in which Mach's principle appears at the cost of modifying Einstein's original theory. It is noted that when Einstein's theory is reformulated by means of the quantities φ and $\gamma_{\mu\nu}$ it takes a form reminiscent of a variety of new field equations proposed by Hoyle (*12*), Yilmaz (*23*), Jordan (*18*), and others, and is strongly similar to the theory of Brans and Dicke (*14*). The important difference is that the field φ is not a new scalar field superimposed on the metric but it is a scalar density related to the metric tensor. In the inertial frame Einstein's theory is Lorentz invariant and hence can also be compared with the reformulations of the gravitational equations in flat space-time in the language of conventional field theory due to Gupta (*24*), Thirring (*17*), and Feynman (*25*).

We conclude with a brief discussion of some remaining unsolved problems connected with Mach's principle.

II. THE PERFECT COSMOLOGICAL PRINCIPLE AND THE BASIC INERTIAL FRAME. BOUNDARY CONDITIONS FOR MACHIAN SOLUTIONS

Our immediate aim is to define the basic inertial frame in General Relativity without which any Machian analysis of inertial effects cannot be given a meaning. Without concerning ourselves with the details of the space-time structure we start from the global system of galaxies in our expanding universe. This cosmological background has a simple structure which conforms to two principles. The first is the so-called Weyl's postulate which states that the world-lines of galaxies diverge from a point in space-time situated in the finite or infinite past. The second is the cosmological principle which expresses the spatial uniformity of the universe so that the distribution of the galaxies would look the same to any observer situated anywhere at a given time. The cosmological principle implies spatial isotropy as well as constancy of curvature throughout space at a definite time.

From the two principles it follows that there exists a preferred system of coordinates (the Robertson-Walker system (*26*)) in which the line element takes the simple form.

$$ds^2 = c^2\,dt^2 - R^2(t)a^2(r)(dx^2 + dy^2 + dz^2) \tag{2.1}$$

where

$$a(r) = (1 + \tfrac{1}{4}kr^2)^{-1} \tag{2.2}$$

and k is the curvature of 3-space.

It has further been shown by Infeld and Schild (*27*) that, for most cosmological models, it is possible to find a coordinate transformation which throws the line element (2.1) into the conformal form

$$ds^2 = \lambda^2(t, r)(c^2\,dt^2 - dx^2 - dy^2 - dz^2). \tag{2.3}$$

This means that the universe as we observe it must be, on the whole, conformally flat. In other words there exists a frame in which, in addition to the cosmological principle and the Weyl postulate being satisfied, we also have light rays travelling along straight lines with speed c. Such coordinate systems will be called conformal cosmological coordinates. These are still not the inertial coordinates we are looking for. In fact, consider a special conformal transformation with a timelike constant acceleration vector. Combining it with a pure Lorentz transformation, we can make the spatial components of the acceleration vanish and this transformation (28) which does not destroy the isotropy of the cosmological background will preserve the general form (2.3) of the line element. The metric (2.3) therefore does not rule out some types of accelerations and if we start from an inertial frame, the new one will be noninertial.

What we need then is a more restrictive form for the cosmological metric which leaves no room for acceleration transformations. This is provided by Bondi and Gold's (*29*) Perfect Cosmological Principle, according to which the universe in the large is uniform not only in space but also in time, so that all points in space-time are equivalent. Since this highly restrictive principle of uniformity has not been disproved by observation so far, we shall adopt it for the purpose of this paper as a rough approximation to the structure of the universe. Such spaces are also conformally flat and in conformal coordinates the line element takes the form

$$ds^2 = \Phi^2(\tau^2)(c^2\,dt^2 - dx^2 - dy^2 - dz^2) = \Phi^2(\tau^2)\eta_{\mu\nu}\,dx^\mu\,dx^\nu, \qquad (2.4)$$

where, as usual, $x^0 = ct$ and $\eta_{\mu\nu}$ is the metric of special relativity with diagonal elements $(1, -1, -1, -1)$. τ^2 is the square length of the position vector defined by

$$\tau^2 = \eta_{\mu\nu}x^\mu x^\nu = c^2t^2 - r^2, \qquad (2.5)$$

and the function Φ, characteristic of a homogeneous space-time is given by

$$\Phi(\tau^2) = \Phi(0)(1 - K^2\tau^2)^{-1}, \qquad (2.6)$$

where K is real for positive and imaginary for negative spatial curvature.

Now the metric (2.4) determines the coordinate system to within a 10-parameter kinematical group. This is the well-known de Sitter group which includes the 6-parameter homogeneous Lorentz group as a subgroup and also includes four displacements which reduce to space-time translations for $K = 0$. Since observationally K is very small, the group of the metric (2.4) is essentially identical with the Poincaré group which is the group of transformation of inertial frames.

The Perfect Cosmological Principle which states that the laws of physics are the same for every observer at any place and any time leads us therefore to a de Sitter structure for the cosmological background and to the invariance of physical laws under the de Sitter group which reduces to the group of Special Relativity for a negligible curvature. This conclusion is in perfect accordance with Fock's insistence (*30*, *31*) that Special Relativity expresses the uniformity of space-time. Fock, however, restricts himself to the case $K = 0$, so that he deals with an empty universe. We simply say, from a Machian standpoint, that Special Relativity (in the generalized sense of de Sitter invariance of physical laws) is a consequence of the approximate uniformity in space and time (the Perfect Cosmological Principle) of the cosmological background.

Having now defined the inertial coordinate system as the one in which the cosmological line element takes the limiting form (2.4) we can state the boundary conditions for the metric $g_{\mu\nu}$ of the actual space-time in this inertial system. We must have asymptotically,

$$g_{\mu\nu} \sim \Phi^2(\tau^2)\eta_{\mu\nu} \tag{2.7}$$

as in (1.1). Introducing the tensor density $\gamma_{\mu\nu}$ with unit determinant by (1.5) and the scalar density φ by (1.3) we can also rewrite the condition (2.7) in the forms (1.2) and (1.4). By definition $\sqrt{-g}$ is a scalar density with weight 1, so that φ is a scalar density of weight $\frac{1}{4}$ and $\gamma_{\mu\nu}$ a tensor density of weight $-\frac{1}{4}$. Expressing deviations of the metric $g_{\mu\nu}$ from the de Sitter metric by means of ξ and $h_{\mu\nu}$ defined by (1.6) and (1.7) respectively, we see that the Lorentz scalar ξ and the Lorentz tensor $h_{\mu\nu}$ obey the boundary conditions

$$\lim_{r\to\infty} \xi = 0, \tag{2.8}$$

and

$$\lim_{r\to\infty} h_{\mu\nu} = 0 \tag{2.9}$$

in the inertial frame.

Furthermore because $\Phi \to 0$ for large r, when t is kept constant, we also have, in the inertial frame,

$$\lim_{r\to\infty} g_{\mu\nu} = 0. \tag{2.10}$$

This last condition was called "degeneration of the metric" by Einstein (*1*) who made an attempt to use it in connection with Mach's principle. However, the special frame in which this degeneration occurs was not specified by Einstein so that it remained as a vague statement which seemed to lead to contradictions. The condition (2.10) is obviously not satisfied in the Robertson-Walker system, nor is it in a system for which $\sqrt{-g} = 1$, this latter being the one favored by Einstein.

In a general noninertial frame, the conformally flat universe is no longer described by one function λ, but by a metric tensor $C_{\mu\nu}$ which satisfies the conditions of conformal flatness. Then, instead of (2.3) we must have the asymptotic metric

$$ds^2 = C_{\mu\nu}\, dx^\mu\, dx^\nu \tag{2.11}$$

where $C_{\mu\nu}$ satisfies the condition

$$R^{\kappa}_{\mu\nu\lambda} = \tfrac{1}{6}R(\delta_\nu^{\ \kappa}C_{\mu\lambda} - \delta_\lambda^{\ \kappa}C_{\mu\nu}) - \tfrac{1}{2}(\delta_\nu^{\ \kappa}R_{\mu\lambda} - \delta_\lambda^{\ \kappa}R_{\mu\nu} + C_{\mu\lambda}R_\nu^{\ \kappa} - C_{\mu\nu}R_\lambda^{\ \kappa}), \tag{2.12}$$

which is the condition for the space to be conformally flat (32). Here the left hand side of (2.12) is the curvature tensor constructed out of the metric $C_{\mu\nu}$. Thus (2.12) is the covariant condition that $g_{\mu\nu}$ must satisfy asymptotically if the cosmological structure is conformal.

On the other hand, if we regard the universe as having constant curvature as a first approximation, then it is a special conformal space-time, that satisfies the more stringent covariant condition

$$R^{\kappa}_{\mu\nu\lambda} = k_0(\delta_\nu{}^\kappa C_{\mu\lambda} - \delta_\lambda{}^\kappa C_{\mu\nu}), \tag{2.13}$$

where k_0 is the curvature of the de Sitter universe. Then (2.12) is automatically satisfied since, as a consequence of (2.13), we also have

$$R^{\kappa}_{\mu\nu\kappa} = R_{\mu\nu} = -3k_0 C_{\mu\nu}\,. \tag{2.14}$$

In virtue of the relation

$$R = -12k_0 \tag{2.15}$$

we can also write (2.13) in the form

$$R^{\kappa}_{\mu\nu\lambda} = -\tfrac{1}{12}R(\delta_\nu{}^\kappa C_{\mu\lambda} - \delta_\lambda{}^\kappa C_{\mu\nu}). \tag{2.16}$$

For weak gravitational fields we can now assume that the deviations of the curvature tensor from the form (2.16) are small. This is a completely covariant condition which replaces (2.7).

In a noninertial frame the de Sitter cosmological structure will be described by a metric $C_{\mu\nu}$ which satisfies (2.16) but not the cosmological postulates. The latter are however satisfied in special frames, namely, the inertial frames for which $C_{\mu\nu}$ takes the form

$$C_{\mu\nu} = \Phi^2(\tau^2)\eta_{\mu\nu}\,. \tag{2.17}$$

These are the frames in which light propagates along straight lines with velocity c. Furthermore in these inertial coordinates the universe looks isotropic while it will generally appear anisotropic in an accelerated system.

In a conformally flat universe where the cosmological principle of spatial homogeneity and Weyl's postulate are valid we can find a coordinate system, which we call the Infeld-Schild system (*27*), in which the cosmological line element takes the form (2.3) with

$$\lambda(t, r) = (1 - K^2\tau^2)^{-1}F\left(\frac{t}{1 - K^2\tau^2}\right) \tag{2.18}$$

instead of the form (2.4) associated with the Perfect Cosmological Principle. Then the Infeld-Schild system will define the inertial frame in the limit in which the function F may be approximated by a constant (the de Sitter limit).

III. SEPARATION OF EINSTEIN'S GRAVITATIONAL EQUATIONS IN AN INERTIAL SYSTEM

In this section we wish to show how the conformal part and the homogeneous de Sitter background of the space-time geometry can be separated in an inertial system of coordinates.

Let the line element be given by

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = \varphi^2 \gamma_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{3.1}$$

where $g_{\mu\nu}$ is the metric tensor, φ the scalar density defined by (1.3), and $\gamma_{\mu\nu}$ the tensor density defined by (1.5). By definition $\sqrt{-g}$ is a scalar density of weight 1. It follows that φ and $\gamma_{\mu\nu}$ have respectively weights ¼ and −½. Define $\mathcal{R}_{\mu\nu}$ as the Ricci tensor constructed out of $\gamma_{\mu\nu}$ and $\mathcal{R}$ as the quantity

$$\mathcal{R} = \gamma^{\mu\nu}\mathcal{R}_{\mu\nu} \tag{3.2}$$

where $\gamma^{\mu\nu}$ is defined by

$$\gamma^{\mu\nu} = (-g)^{1/4} g^{\mu\nu} = \varphi^2 g^{\mu\nu}, \tag{3.3}$$

so that we have

$$\gamma^{\mu\kappa}\gamma_{\kappa\nu} = \delta_\nu^\mu. \tag{3.4}$$

Then, according to a standard formula of Riemannian geometry (see ref. *32*, p. 90) we have

$$\begin{aligned} R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = \mathcal{R}_{\mu\nu} - \tfrac{1}{2}\gamma_{\mu\nu}\mathcal{R} - 4\varphi^{-2}[(\partial_\mu\varphi)(\partial_\nu\varphi) - \tfrac{1}{4}\gamma_{\mu\nu}(\partial_\lambda\varphi)(\partial^\lambda\varphi)] \\ + 2\varphi^{-1}[(\partial_\mu\varphi)_{;\nu} - \gamma_{\mu\nu}\Box_{(\gamma)}\varphi], \end{aligned} \tag{3.5}$$

where the covariant derivative on the right hand side refers to the tensor $\gamma_{\mu\nu}$ and $\Box_{(\gamma)}$ is the generalized D'Alembert operator constructed with $\gamma_{\mu\nu}$, so that

$$\Box_{(\gamma)}\varphi = (\varphi_{,\nu})^{;\nu} = (1/\sqrt{-\gamma})\partial_\mu(\sqrt{-\gamma}\gamma^{\mu\nu}\partial_\nu\varphi). \tag{3.6}$$

Remembering from (1.5) that

$$\gamma = \operatorname{Det}\|\gamma_{\mu\nu}\| = -1,$$

we find

$$\Box_{(\gamma)}\varphi = \partial_\mu(\gamma^{\mu\nu}\partial_\nu\varphi). \tag{3.7}$$

With the customary definition of $R_{\mu\nu}$ as

$$R_{\mu\nu} = \partial_\mu\partial_\nu \log\sqrt{-g} - \Gamma^\alpha_{\mu\nu}\partial_\alpha \log\sqrt{-g} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\alpha\nu} - \partial_\alpha\Gamma^\alpha_{\mu\nu}, \tag{3.8}$$

Einstein's covariant equations of gravitation read

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu}. \tag{3.9}$$

In Einstein's theory κ is positive and proportional to the gravitational constant, the factor of proportionality being $8\pi/c^4$. Here we shall not specify κ as yet. Defining $\mathcal{R}_\mu{}^\nu$ as

$$\mathcal{R}_\mu{}^\nu = \gamma^{\nu\lambda}\mathcal{R}_{\mu\lambda} \tag{3.10}$$

by means of the contravariant densities $\gamma^{\nu\lambda}$ defined by (3.4) we find, using (3.5),

$$\begin{aligned}\mathcal{R}_\mu{}^\nu - \tfrac{1}{2}\delta_\mu{}^\nu\mathcal{R} - 4\varphi^{-2}[(\partial_\mu\varphi)(\partial^\nu\varphi) - \tfrac{1}{4}\delta_\mu{}^\nu(\partial_\lambda\varphi)(\partial^\lambda\varphi)]\\ + 2\varphi^{-1}[(\partial_\mu\varphi)^{;\nu} - \delta_\mu{}^\nu\Box_{(\gamma)}\varphi] = -\kappa\varphi^2 g^{\nu\lambda}T_{\mu\lambda}\,.\end{aligned} \tag{3.11}$$

Now we introduce the tensor density $\mathfrak{T}_\mu{}^\nu$ of weight $\tfrac{3}{4}$ through the definition:

$$\mathfrak{T}_\mu{}^\nu = \varphi^3 g^{\nu\lambda}T_{\mu\lambda}\,. \tag{3.12}$$

The tensor density equation (3.11) takes the form

$$\mathcal{R}_\mu{}^\nu - \tfrac{1}{2}\delta_\mu{}^\nu\mathcal{R} = -\kappa\varphi^{-1}\mathfrak{T}_\mu{}^\nu + \mathfrak{M}_\mu{}^\nu(\varphi), \tag{3.13}$$

where

$$\begin{aligned}\mathfrak{M}_\mu{}^\nu(\varphi) = 4\varphi^{-2}[(\partial_\mu\varphi)(\partial^\nu\varphi) - \tfrac{1}{4}\delta_\mu{}^\nu(\partial_\lambda\varphi)(\partial^\lambda\varphi)]\\ - 2\varphi^{-1}[(\partial_\mu\varphi)^{;\nu} - \delta_\mu{}^\nu\Box_{(\gamma)}\varphi],\end{aligned} \tag{3.14}$$

the indices being raised by means of the tensor density $\gamma^{\nu\lambda}$.

The introduction of the density $\mathfrak{T}_\mu{}^\nu$ can be justified by a straightforward generalization of the expression of the energy momentum tensor for a system of mass points in flat space. In flat space we have (see for example ref. *17*),

$$t_\mu{}^\nu = \sum_i m_i c^2 \int du_i\, \delta^{(4)}(x - z_i(u_i))\, \frac{dz_i{}^\nu}{du_i}\frac{dz_i{}^\lambda}{du_i}\,\eta_{\lambda\mu} \tag{3.15}$$

where

$$du^2 = \eta_{\lambda\mu}\, dz^\lambda\, dz^\mu \tag{3.16}$$

is the line element in flat space. In the general Riemannian space with metric given by (3.1) we introduce the scalar density $d\tau$ of weight $-\tfrac{1}{4}$ defined by

$$d\tau^2 = \gamma_{\mu\nu}\, dx^\mu\, dx^\nu \tag{3.17}$$

so that we have

$$ds = \varphi\, d\tau \tag{3.18}$$

for the scalar line element. Now, in an inertial system we have $\gamma_{\mu\nu} \to \eta_{\mu\nu}$ asymptotically, so that we also have

$$d\tau \to du. \tag{3.19}$$

The tensor density obtained from (3.15) by the substitutions of $d\tau$ for du and $\gamma_{\mu\nu}$ for $\eta_{\mu\nu}$, namely,

$$\mathfrak{Z}_\mu^{\ \nu} = \sum_i m_i c^2 \int d\tau_i \, \delta^{(4)}(x - z_i(\tau_i)) \frac{dz_i^{\ \nu}}{d\tau_i} \frac{dz_i^{\ \lambda}}{d\tau_i} \gamma_{\lambda\mu} \tag{3.20}$$

will be a tensor density of weight $\frac{3}{4}$, which asymptotically goes over to the cartesian energy momentum tensor of flat space. This is then the explicit form for a system of mass points of the tensor density defined by (3.12). Equations (3.13) and (3.14) together with the condition on the determinant (1.5) are equivalent to the original Einstein equations (3.9) and form the starting of our investigation.

Contracting Eq. (3.13) we also find

$$-\mathfrak{R} = -\kappa\varphi^{-1}\mathfrak{Z} + \mathfrak{M}_\nu^{\ \nu} = \varphi^{-1}(-\kappa\mathfrak{Z} + 6\Box_{(\gamma)}\varphi). \tag{3.21}$$

The geodesic equation which describes the motion of a test particle is

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \tag{3.22}$$

where $\Gamma^\lambda_{\mu\nu}$ is the Christoffel symbol corresponding to the tensor $g_{\mu\nu}$ that appears in the metric (3.1). By means of (3.17) and (3.18) the geodesic equation takes the well-known form

$$\frac{d}{d\tau}\left(\varphi \frac{dx^\lambda}{d\tau}\right) = \gamma^{\lambda\rho}\partial_\rho\varphi - \varphi \left\{ \begin{matrix} \lambda \\ \mu\nu \end{matrix} \right\} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}, \tag{3.23}$$

where the Christoffel symbol is constructed with the help of the tensor density $\gamma_{\mu\nu}$. An alternative form is

$$\frac{d}{d\tau}\left(\varphi\gamma_{\sigma\nu} \frac{dx^\nu}{d\tau}\right) = \partial_\sigma\varphi + \tfrac{1}{2}\varphi(\partial_\sigma\gamma_{\lambda\nu}) \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau}. \tag{3.24}$$

The quantities ξ and $h_{\mu\nu}$ introduced by (1.6) and (1.7) are small in an inertial system away from sources of gravitational fields. Let us first consider the case of vanishing ξ and $h_{\mu\nu}$. Then, using (2.6), we can write

$$\varphi \cong \Phi(\tau^2) = \Phi(0)(1 - \tau^2/4R^2)^{-1} \tag{3.25}$$

where R is the radius of curvature of the de Sitter universe. We find for the tensor density defined in (3.14)

$$\mathfrak{M}_\mu^{\ \nu}(\varphi) \cong \mathfrak{M}_\mu^{\ \nu}(\Phi) = \frac{3}{\Phi^2(0)R^2} \Phi^2\delta_\mu^{\ \nu}. \tag{3.26}$$

In the general case we may put

$$\alpha_{\mu\nu} = [1 + (\xi/\Phi)]^2\gamma_{\mu\nu} = [1 + (\xi/\Phi)]^2(\eta_{\mu\nu} + h_{\mu\nu}) \tag{3.27}$$

so that $\alpha_{\mu\nu}$ differs little from $\eta_{\mu\nu}$ away from gravitational sources and the metric takes the form

$$g_{\mu\nu} = \Phi^2(\tau^2)\alpha_{\mu\nu} \tag{3.28}$$

in the inertial coordinate system.

If $\bar{\mathcal{R}}_\mu{}^\nu$ is the Ricci tensor corresponding to the metric $\alpha_{\mu\nu}$, we obtain, similarly to (3.13),

$$\bar{\mathcal{R}}_\mu{}^\nu - \tfrac{1}{2}\delta_\mu{}^\nu\bar{\mathcal{R}} = -\kappa\Phi^{-1}\bar{\mathfrak{J}}_\mu{}^\nu + \mathfrak{M}_\mu{}^\nu(\Phi), \tag{3.29}$$

where $\bar{\mathfrak{J}}_\mu{}^\nu$ is given by

$$\bar{\mathfrak{J}}_\mu{}^\nu = \Phi^3 g^{\nu\lambda}T_{\mu\lambda} \tag{3.30}$$

and $\mathfrak{M}_\mu{}^\nu(\Phi)$ takes the limiting value (3.26) when $\alpha_{\mu\nu}$ is approximated by $\eta_{\mu\nu}$. The contracted form of (3.29) is

$$-\bar{\mathcal{R}} = \Phi^{-1}(-\kappa\bar{\mathfrak{J}} + 6\Box_{(\gamma)}\Phi). \tag{3.31}$$

On the other hand we have

$$\Box\Phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\Phi = (2/\Phi^2(0)R^2)\Phi^3, \tag{3.32}$$

so that in the zero approximation ($\xi = 0$, $h_{\mu\nu} = 0$) we have

$$\tfrac{1}{6}\kappa\bar{\mathfrak{J}} \cong \Box\Phi = (2/\Phi^2(0)R^2)\Phi^3. \tag{3.33}$$

Introducing the scalar T through

$$T = g^{\mu\nu}T_{\mu\nu}, \tag{3.34}$$

the contracted form of (3.30) reads

$$T = \Phi^{-3}\bar{\mathfrak{J}} \tag{3.35}$$

so that (3.33) takes the form

$$\tfrac{1}{6}\kappa T = 2/\Phi^2(0)R^2. \tag{3.36}$$

For small deviations from the de Sitter metric we may put

$$\alpha_{\mu\nu} = \eta_{\mu\nu} + a_{\mu\nu}, \tag{3.37}$$

where $a_{\mu\nu}$ is small in an inertial system away from inhomogeneities. We have

$$a_{\mu\nu} = (2\xi/\Phi)\eta_{\mu\nu} \tag{3.38}$$

Then, in the zero approximation ($a_{\mu\nu} = 0$), the parameters of the de Sitter background are related to the trace of the energy-momentum tensor by means of (3.36). The field $\alpha_{\mu\nu}$ is then determined from the equations (3.29).

We note that the equation of motion (3.23) of a test particle may also be rewritten in terms of Φ and $\alpha_{\mu\nu}$ in the form

$$\frac{d}{d\bar{\tau}}\left(\Phi \frac{dx^\lambda}{d\bar{\tau}}\right) = \alpha^{\lambda\rho}\partial_\rho\Phi - \Phi\overline{\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}}\frac{dx^\mu}{d\bar{\tau}}\frac{dx^\nu}{d\bar{\tau}}, \tag{3.39}$$

where

$$d\bar{\tau}^2 = \alpha_{\mu\nu}\, dx^\mu\, dx^\nu,$$

and the Christoffel symbol refers to the metric $\alpha_{\mu\nu}$.

IV. THE DE SITTER BACKGROUND

First we consider the case of a homogeneous universe where the functions $a_{\mu\nu}$ describing the inhomogeneities vanish. For $\kappa > 0$ we find, from (3.36)

$$\Phi(\tau^2) = \frac{1}{R}\left(\frac{\kappa T}{12}\right)^{-1/2}\left(1 + \frac{r^2 - c^2t^2}{4R^2}\right)^{-1} \tag{4.1}$$

as the solution of (3.33). If $\kappa < 0$, we have

$$\Phi'(\tau^2) = \frac{1}{R}\left(\frac{-\kappa T}{12}\right)^{-1/2}\left(1 - \frac{r^2 - c^2t^2}{4R^2}\right)^{-1}. \tag{4.2}$$

In the first case the de Sitter universe has positive spatial curvature for $t = 0$ while the second case corresponds to negative curvature. In these solutions the radius of curvature R is arbitrary. We now propose to define the total mass of a de Sitter universe in the case of positive curvature. In the de Sitter case, we have, from (3.21)

$$\Box\Phi = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi = \frac{\kappa}{6}\mathfrak{J}, \tag{4.3}$$

where, in virtue of (3.20),

$$\mathfrak{J} = \sum_i m_i c^2 \int du_i\, \delta^{(4)}(x - z_i(u_i)). \tag{4.4}$$

On the other hand, for a static distribution of matter we have the Poisson equation

$$\nabla^2 V = 4\pi\kappa' \sum_i m_i \delta(\mathbf{r} - \mathbf{r}_i), \tag{4.5}$$

where V is the Newtonian potential. The total mass of the system is in this case

$$M = \sum_i m_i = \frac{1}{4\pi\kappa'}\int (\nabla^2 V)\, d^3\mathbf{r}. \tag{4.6}$$

We see that (4.3), in the inertial system, is the special relativistic generalization of (4.5), Φ playing the role of a nonstatic Newtonian potential. We may then define a total mass enclosed in a de Sitter universe at time t by

$$M(t) = -\frac{6}{\kappa c^2}\int\left(\nabla^2 - \frac{\partial^2}{c^2\partial t^2}\right)\Phi\, d^3\mathbf{r} = \frac{6}{\kappa c^2}\int \Box\Phi\, d^3\mathbf{r}. \tag{4.7}$$

Using (3.32) and (3.36) we find

$$M(t) = \frac{12}{\kappa c^2}\frac{1}{\Phi^2(0)R^2}\int \Phi^3\, d^3\mathbf{r} = \frac{1}{c^2}\,T\int \Phi^3\, d^3\mathbf{r}, \tag{4.8}$$

so that, in the case of positive curvature

$$\begin{aligned} M(t) &= \frac{1}{c^2}\,T\Phi^3(0)\int\left(1+\frac{r^2-c^2t^2}{4R^2}\right)^{-3} d^3\mathbf{r} \\ &= \frac{4\pi}{c^2}\,T\Phi^3(0)\int_0^\infty\left(1+\frac{r^2-c^2t^2}{4R^2}\right)^{-3} r^2\, dr. \end{aligned} \tag{4.9}$$

Carrying out the integration, we obtain

$$M(t) = \frac{2\pi^2}{c^2}\,TR^3\Phi^3(0)\left(1-\frac{c^2t^2}{4R^2}\right)^{-3/2}, \tag{4.10}$$

or

$$M(t) = M(0)\left(1-\frac{c^2t^2}{4R^2}\right)^{-3/2} \tag{4.11}$$

with

$$M(0) = (2\pi^2/c^2)TR^3\Phi^3(0) = (24\pi^2/\kappa c^2)R\Phi(0). \tag{4.12}$$

Thus we have the important relation

$$\frac{R}{M(0)} = \frac{1}{24\pi^2}\frac{\kappa c^2}{\Phi(0)} \qquad (\kappa > 0), \tag{4.13}$$

which expresses the ratio $\kappa/\Phi(0)$ in terms of the total mass and radius of the de Sitter universe.

For the mass density $\mathfrak{z}$ in the de Sitter universe we find, from (3.35)

$$\mathfrak{z} = \bar{\mathfrak{z}} = T\Phi^3, \tag{4.14}$$

so that, using (4.12) we find

$$\mathfrak{z}(0) = T\Phi^3(0) = \frac{1}{2\pi^2}\frac{M(0)c^2}{R^3}. \tag{4.15}$$

This is just the total rest energy of the de Sitter universe divided by its total volume.

Coming now to the case of negative curvature we note that in this case the integral on the right hand side of (4.9) diverges and the enclosed mass is infinite. A relation like (4.13) connecting the constant κ with the mass of the universe no longer exists for the spatially open de Sitter world.

We owe another word of explanation in connection with the statement that the de Sitter universe is empty. What is meant by emptiness in the literature is the absence of a model of a fluid composed of stable particles to describe the properties of the de Sitter universe and generate its metric. This, however, does not mean that in such a universe the energy-momentum density of matter vanishes identically. From (3.29) and (3.26) we find

$$\kappa \mathfrak{I}_{\mu\nu} = (3/\Phi^2(0)R^2)\Phi^3\eta_{\mu\nu} . \tag{4.16}$$

A distribution of stable matter cannot have an energy-momentum distribution given by (4.16). It cannot represent a radiation filled universe either, since $\mathfrak{I} \neq 0$.

It is possible to find a physical interpretation of an energy-momentum tensor of the form (4.16) if we think of the curvature induced by a massive particle appearing and immediately disappearing at a world point with coordinates ξ_μ, that is, at the point with position vector $\boldsymbol{\xi}$ at time ξ_0. Such a single event may be called a "mass scintillation." The scalar density Φ then satisfies an equation of the form

$$\Box \Phi_\xi = \gamma \delta^{(4)}(x - \xi), \tag{4.17}$$

where γ is a constant. One solution of this inhomogeneous equation is

$$\begin{aligned} \Phi_\xi &= (4\pi)^{-1}\gamma\delta(|x - \xi|^2) \quad \text{for } |x - \xi|^2 > 0, \\ \Phi_\xi &= 0 \qquad\qquad\qquad\qquad\quad \text{for } |x - \xi|^2 < 0. \end{aligned} \tag{4.18}$$

It can now be shown that the energy-momentum tensor density corresponding to such a mass scintillation is proportional to $\eta_{\mu\nu}$ so that the expression $\mathfrak{I}_{\mu\nu}$ given by (4.16) may be regarded as arising from a uniform distribution of mass scintillations.

The motion of a test particle in the de Sitter universe with positive spatial curvature is given by

$$\frac{d}{du}\left(\Phi \frac{dx^\mu}{du}\right) = \eta^{\mu\nu}\partial_\nu \Phi = \frac{x^\mu}{2\Phi(0)R^2}\Phi^2, \tag{4.19}$$

or

$$\frac{d}{du}\left(\frac{\Phi}{\Phi(0)}\frac{dx^\mu}{du}\right) = \frac{x^\mu}{2R^2}\left(\frac{\Phi}{\Phi(0)}\right)^2. \tag{4.20}$$

Neglecting R^{-4} we find

$$\frac{d^2\mathbf{x}}{du^2} \cong \frac{\mathbf{x}}{2R^2} \tag{4.21}$$

or

$$\frac{d\mathbf{x}}{dt} \cong \pm\frac{c\mathbf{x}}{\sqrt{2}R} = h\mathbf{x}. \tag{4.22}$$

For the time interval $(-2R/c)$ to 0 we have the positive sign corresponding to an expansion with Hubble's constant

$$h = c/R\sqrt{2}. \tag{4.23}$$

From (4.19) we see that $\Phi(0)$ is proportional to the inertial mass of the particle. When the total mass of the universe decreases, R being kept constant, it follows from (4.13) that $\Phi(0)$, and hence the inertia of the particle, decreases too. This is in accordance with Mach's principle as it will be shown in the following sections in more detail.

V. REPULSIVE COSMIC FORCES IN CONFORMAL SPACE-TIME. IMPOSSIBILITY OF A SCALAR THEORY OF GRAVITATION

In this section we shall treat a space-time model which is conformally flat but not homogeneous. It is therefore less simple than the de Sitter case. It does not satisfy Bondi and Gold's Perfect Cosmological Principle but the more restricted form of the cosmological principle associated with spatial homogeneity. We start from a de Sitter background due to a uniform spread of mass scintillations. On this background we superimpose one inhomogeneity, keeping however the conformally flat character of the geometry. Let us take the stable mass point as the origin of coordinates. The de Sitter background allows us to define an inertial system so that we can use the equations of Section III with the condition

$$h_{\mu\nu} = 0, \quad \text{or} \quad \gamma_{\mu\nu} = \eta_{\mu\nu}\,, \tag{5.1}$$

which expresses conformal flatness. We have

$$ds^2 = \varphi^2\,\eta_{\mu\nu}\,dx^\mu\,dx^\nu, \tag{5.2}$$

with

$$\varphi = \Phi(\tau^2) + \xi(r). \tag{5.3}$$

This metric describes a static isotropic field due to the inhomogeneity at the origin. To determine ξ we use Eq. (3.21) which takes the form

$$\Box\varphi = \Box(\Phi + \xi) = \tfrac{1}{6}\kappa\mathfrak{J}. \tag{5.4}$$

Now $\mathfrak{J}$ has two parts, one coming from the de Sitter structure given by (3.33) and the other from the existence of the stable mass point at the origin. Thus we have

$$\mathfrak{J} = \frac{12}{\kappa} \frac{\Phi^3}{\Phi^2(0)R^2} + \mu c^2 \delta(\mathbf{r}), \tag{5.5}$$

where m is the mass of the inhomogeneity.

With this value of $\mathfrak{J}$, (5.4) gives the following equation for ξ

$$\nabla^2 \xi = -\tfrac{1}{6}\kappa\mu c^2 \delta(\mathbf{r}). \tag{5.6}$$

Since in the inertial system we require the metric to go over to the de Sitter metric away from inhomogeneities we must solve (5.6) with the Machian condition

$$\lim_{r\to\infty} \xi = 0. \tag{5.7}$$

Remembering that κ is positive, this solution is therefore

$$\xi = \frac{\kappa c^2}{24\pi} \frac{\mu}{r}. \tag{5.8}$$

The metric of this universe has then the form

$$ds^2 = \left(\Phi + \frac{\kappa c^2}{24\pi} \frac{\mu}{r}\right)^2 du^2, \tag{5.9}$$

where du^2 is the Minkowski line element given by (3.16).

Let us now discuss the motion of a test particle in this universe. Equation (3.23) gives

$$\frac{d}{du}\left[\left(\frac{\Phi}{\Phi(0)} + \frac{\kappa c^2}{24\pi\Phi(0)} \frac{\mu}{r}\right) \frac{dx^\mu}{du}\right] = \frac{x^\mu}{2R^2}\left(\frac{\Phi}{\Phi(0)}\right)^2 + \frac{\kappa c^2}{24\pi\Phi(0)} \eta^{\mu\nu}\partial_\nu\left(\frac{\mu}{r}\right). \tag{5.10}$$

Neglecting for the moment the cosmical expansion, and using (4.13), we find approximately

$$\frac{d}{du}\left[\left(1 + \frac{R}{\pi M(0)} \frac{\mu}{r}\right) \frac{d\mathbf{x}}{du}\right] = -\frac{R}{\pi M(0)} \nabla\left(\frac{\mu}{r}\right). \tag{5.11}$$

Thus, the test particle will be subject to a repulsive radial force proportional to the mass μ of the body at the origin, the coefficient of proportionality being

$$C = Rc^2/\pi M(0), \tag{5.12}$$

where R is the radius of the de Sitter universe and $M(0)$ its total mass at time $t = 0$.

This is not a gravitational, but rather an antigravitational force acting between massive bodies in a conformal space-time.

Supposing that more masses m_i are piled up at points with coordinates $\mathbf{a}_i$, the geometry being kept conformally flat, these various points will repel each other with a force inversely proportional to the square of their relative distances. If we take these massive inhomogeneities in a conformal space-time as the galaxies we see that we arrive at a picture of the universe in which galaxies exert on each other a Coulomb-like repulsive force. This is exactly like the "electric universe" model of Bondi and Lyttleton (*33*). These authors have postulated such a repulsive force between galaxies to which, however, they ascribe an electric origin. They show that, as a result, one gets an expanding universe obeying Hubble's law. The repulsive force needed between two galaxies corresponds to a charge excess of the order of 10^{-18}, so that the force is of the order

$$f \cong 10^{-36}\, e^2 \frac{\mu^2}{m_0^2} \frac{1}{r^2}, \tag{5.13}$$

where μ is the mass of a galaxy and m_0 the mass of a hydrogen atom. Since we have numerically

$$e^2/Gm_0^2 \cong 10^{36}, \tag{5.14}$$

G being the gravitational constant, we find

$$f \cong G\mu^2/r^2, \tag{5.15}$$

so that one needs a repulsive force of the order of the gravitational force. This is exactly the repulsion given by our Machian model provided the constant C of (5.12) is of the order of the gravitational constant.

Now consider our model of a conformal space-time in which N galaxies of mass μ are situated at the positions $\mathbf{a}_i$ at time $t = 0$.

The total mass of the universe is given by

$$\mathfrak{M}(0) = \frac{6}{\kappa c^2} \int (\Box\varphi)_{t=0}\, d^3\mathbf{r} = M(0) - \frac{6}{\kappa c^2} \int \nabla^2\xi\, d\mathbf{r} = M(0) + N\mu. \tag{5.16}$$

The total mass comes partly from the mass scintillations of the de Sitter background and partly from the mass of stable matter. The equation for ξ is

$$\nabla^2\xi = -\tfrac{1}{6}\kappa c^2 \mu \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{a}_i), \tag{5.17}$$

so that the solution vanishing at infinity gives

$$\varphi = \Phi + \xi = \Phi(\lambda^2) + \frac{\kappa\mu c^2}{24\pi} \sum_i \frac{1}{|\mathbf{r} - \mathbf{a}_i|}, \tag{5.18}$$

or, using (4.13),

$$\varphi_0 = (\varphi)_{t=0} = \frac{\kappa c^2}{24\pi}\left(\frac{M(0)}{\pi R}\frac{1}{1 + r^2/(4R^2)} + \mu \sum_i \frac{1}{|\mathbf{r} - \mathbf{a}_i|}\right). \tag{5.19}$$

Let us introduce the function

$$V(r) = \mu \sum_i \frac{1}{|\mathbf{r} - \mathbf{a}_i|}, \tag{5.20}$$

assuming that galaxies are spread uniformly within a sphere of radius A. Since the total mass of galaxies is $N\mu$, we find from potential theory

$$V(r)\begin{cases} = \dfrac{3N\mu}{2A}\left(1 - \dfrac{r^2}{3A^2}\right) & \text{for } r \leqq A, \\ = \dfrac{N\mu}{r} & \text{for } r \geqq A. \end{cases} \tag{5.21}$$

This is a decreasing function of r similar in behavior to the function $(1 + r^2/4R^2)^{-1}$ which occurs in the first term of (5.19). Thus in the static approximation the galaxies contribute to the general curvature and total mass of the spatially closed universe. Since the force between these galaxies has been shown to be repulsive in a conformal space-time, the radius A characterizing the distribution of stable matter will be an increasing function of time. The function φ in a conformal universe filled with stable matter will then have the form $\varphi(t, r)$ instead of the form $\varphi(\tau^2)$ in the de Sitter universe.

In the approximation of neglecting the square of the curvature we find that at time $t = 0$, φ_0 is approximated by the following constant

$$\varphi_0 \cong \frac{\kappa c^2}{24\pi}\left(\frac{M(0)}{\pi R} + \frac{3N\mu}{2A}\right) = \frac{\kappa c^2}{24\pi}\frac{M'}{R'}, \tag{5.22}$$

where

$$M' = M(0) + N\mu \tag{5.23}$$

is the total mass of the universe and R' an average radius of curvature defined by (5.22).

Summing up the discussion in this section, we can state that the Machian boundary conditions rule out a gravitational theory based on a scalar field in a conformally flat universe. It is known that without Mach's principle there is no a priori reason against a scalar theory of gravitation (*17*). On the other hand if the universe at large is conformally flat, the Machian solutions for the metric impose repulsive forces of the order of gravitational forces (antigravitational forces) between galaxies that cause the galactic system to expand in accordance with Hubble's law against a de Sitter background.

VI. TENSOR THEORY OF GRAVITATION. SIGN AND MAGNITUDE OF THE GRAVITATIONAL CONSTANT

We have seen in the preceding section that if the universe is conformally flat in the first approximation, as it seems to be observationally, then Machian boundary conditions allow only repulsive forces in this cosmological background. It is our purpose to show in this section that attractive gravitational interactions are automatically introduced if the metric is not conformally flat, through the "tensorial" part $h_{\mu\nu}$ of the metric and, further, that the strength of this interaction is independent of the constant κ that appears in Einstein's fundamental equation (3.9), but is determined by the distribution of matter of the conformal approximation, that is, by the constants referring to the cosmological background.

Let us now consider the equation (3.13) in the case of a body at the origin which destroys the conformal character of the cosmological background. We have from (3.20)

$$\mathfrak{T}_4^{\ 4} = mc^2\delta(\mathbf{r}), \tag{6.1}$$

other components of $\mathfrak{T}_\mu^{\ \nu}$ being taken as zero. To start with we take the approximation in which the square of the curvature is neglected. We find then, using (5.22)

$$\varphi(t, r) \cong \varphi_0 , \tag{6.2}$$

where

$$\varphi_0 = \frac{\kappa c^2}{24\pi^2} \frac{M(0)}{R} \tag{6.3}$$

for a de Sitter universe and

$$\varphi_0 = \frac{\kappa c^2}{24\pi} \frac{M'}{R'} = \frac{\kappa c^2}{24\pi} \left(\frac{M(0)}{\pi R} + \frac{3N\mu}{2A} \right) \tag{6.4}$$

in a model of N galaxies distributed uniformly within a sphere of radius A against a de Sitter background.

On the other hand the tensor density $\mathfrak{M}_\mu^{\ \nu}$ (3.14) associated with the cosmological structure is of order R^{-2}. Hence in our approximation it is negligible. Equation (3.13) then takes the form

$$\mathfrak{R}_\mu^{\ \nu} - \frac{1}{2}\,\delta_\mu^{\ \nu}\mathfrak{R} \cong -\kappa\varphi_0^{-1}\,\mathfrak{T}_\mu^{\ \nu} \cong -\frac{24\pi}{c^2}\frac{R'}{M'}\,\mathfrak{T}_\mu^{\ \nu}. \tag{6.5}$$

We note that in this equation κ has disappeared. Further, the tensor density $\gamma_{\mu\nu}$ out of which the Ricci tensor density is constructed satisfies the condition

$$\sqrt{-\gamma} = 1 \qquad (\gamma = \mathrm{Det}\,\|\gamma_{\mu\nu}\|). \tag{6.6}$$

The tensor density equation (6.5) has the same form as the standard Einstein equations with a positive gravitation constant (see for instance ref. *34*, p. 179). The static solution satisfying conditions (1.4) and (6.6) and having spherical symmetry for a point mass m at the origin is the well-known de Sitter solution (ref. *34*, p. 202)

$$\gamma_{00} = 1 - \frac{\alpha m}{r}, \quad \gamma_{0s} = 0,$$
$$\gamma_{rs} = -\delta_{rs} - \frac{\alpha m}{r} \frac{1}{1 - (\alpha m/r)} \frac{x_r\, x_s}{r^2}, \tag{6.7}$$

where

$$\alpha = 6R'/M'. \tag{6.8}$$

For the weak fields $h_{\mu\nu}$ defined by (3.27) we have

$$h_{00} = -\frac{\alpha m}{r}, \qquad h_{rs} = -\frac{\alpha m}{r} \frac{x_r\, x_s}{r^2}, \tag{6.9}$$

so that

$$h_\mu^\mu = h_{00} - h_{rr} = 0. \tag{6.10}$$

The effective gravitational constant is given by

$$G = \tfrac{1}{2}c^2\alpha = 3c^2\alpha R'/M'. \tag{6.11}$$

In the case of a pure de Sitter background we have

$$G \cong 3\pi c^2 R/M(0), \tag{6.12}$$

and in the case of N galaxies against a de Sitter background we have (6.11) with R'/M' given by (5.22), that is

$$G \cong 3c^2 \left(\frac{M(0)}{\pi R} + \frac{3N\mu}{2A}\right)^{-1}, \tag{6.13}$$

or, if the contribution of mass scintillations could be neglected besides the contribution of stable galaxies

$$G \cong 2c^2 A/(N\mu), \tag{6.14}$$

where A is the radius of space enclosing N galaxies of mass μ each.

If R^{-2} or A^{-2} are not neglected we see that G must be space and time dependent.

The geodesic equation of motion for a test particle moving in the field of a mass point against a conformally flat uniform cosmological background takes the approximate form

$$\frac{d}{d\tau}\left(\varphi_0 \frac{dx^\lambda}{d\tau}\right) \cong -\varphi_0 \left\{ {\lambda \atop \mu\nu} \right\} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \tag{6.15}$$

in the approximation of neglecting the square of the curvature. In the non-relativistic approximation we obtain

$$\frac{d^2x^s}{dt^2} \cong -\frac{1}{2}c^2\partial_s\gamma_{00} \cong -\frac{Gm}{r^2}\frac{x^s}{r} \cong -3\frac{c^2R'm}{M'}\frac{x^s}{r^3}. \tag{6.16}$$

Thus, the gravitational force on a test particle due to the tensor density $\gamma_{\mu\nu}$ is attractive with a gravitational constant given by (6.11).

In an empty universe the masses $M(0)$ and μ vanish so that we have

$$\varphi_0 = 0, \tag{6.17}$$

and the equation of motion (6.15) is satisfied when the velocity is an arbitrary function of the time. Thus, Newton's law of inertia is not satisfied in an empty universe, in accordance with Mach's principle. Equation (6.15) shows that the inertial mass of a test particle is proportional to φ_0 and hence to M'/R'. We may then put

$$\bar{m}_{\text{in}} = (M'/3c^2R')\,\bar{m} = G^{-1}\bar{m} \tag{6.18}$$

if m is the gravitational mass of the test particle. Then the gravitational force acting on the test particle becomes

$$f^s_{\text{grav}} = -\bar{m}_{\text{in}}\frac{Gm}{r^2}\frac{x^s}{r} = -\frac{\bar{m}m}{r^2}\frac{x^s}{r}, \tag{6.19}$$

where m is the gravitational mass of the source and r its distance to the test particle. Formula (6.18) shows that the inertial mass is proportional to the total mass contained in the universe and inversely proportional to its radius so that in general it can be space and time dependent. For a discussion of Eq. (6.18) the reader is referred to ref. *10*.

In the case of a pure de Sitter background the tensor density $\mathfrak{M}_{\mu\nu}$ given by (3.14) is approximately proportional to the metric $\gamma_{\mu\nu}$, the factor of proportionality being $3/R^2$, so that instead of the Schwarzschild solution we must take the solution (see for instance ref. *35*)

$$\gamma_{00} = \left(1 - \frac{\alpha m}{r} - \frac{r^2}{R^2}\right), \qquad \gamma_{0s} = 0,$$
$$\gamma_{rs} = -\delta_{rs} - \left(\frac{\alpha m}{r} + \frac{r^2}{R^2}\right)x_r x_s \Big/ r^2\left(1 - \frac{\alpha m}{r} - \frac{r^2}{R^2}\right). \tag{6.20}$$

Finally, we may note that the attractive gravitational constant G given by (6.17) in a de Sitter universe and associated with the tensor density $\gamma_{\mu\nu}$ is of the same order as the constant of cosmic repulsive forces C given by (5.12) and associated with the scalar density φ. This remark justifies our comments in

Section V on the fact that the expansion of the universe due to these repulsive forces has the correct order of magnitude. Another remark concerns the incompatibility of the spatially infinite de Sitter universe with Mach's principle since in that case the gravitational constant would vanish because $M(0)$ would diverge (4).

VII. INERTIAL FORCES GENERATED BY UNIFORM ACCELERATION

In this section we show that, in accordance with Mach's principle, the inertial force acting on a uniformly accelerated body may be interpreted as the cosmic repulsive force due to the effect of the uniform substratum moving with the opposite acceleration with respect to the test body. Thus, in this case, the complete relativity of motion can be demonstrated. The notion of inertial force then becomes superfluous. To simplify the discussion we take a cosmological background which is a pure de Sitter universe. The same arguments are also valid for a conformally flat, spatially homogeneous universe such as the model discussed in Section V. In the inertial system which is defined up to a de Sitter transformation the metric is given by (2.4) where $\Phi(\tau^2)$ satisfies

$$\Phi = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi = \frac{\kappa}{6}\mathfrak{J} = \frac{\kappa}{6}T\Phi^3(\tau^2), \tag{7.1}$$

in virtue of (4.3) and (4.14).

Now, suppose that the source $\mathfrak{J}$ of the field Φ is uniformly accelerated with respect to the inertial frame. This is done by subjecting the coordinates to a conformal transformation corresponding to an acceleration $\mathbf{a}$. Space-time still remains conformal in structure, but is no longer uniform. The flat space line element transforms according to the law

$$du = \lambda(t', \mathbf{r}')du', \tag{7.2}$$

where

$$\lambda(t', \mathbf{r}') = \left[1 - \frac{\mathbf{a}\cdot\mathbf{r}'}{c^2} + \frac{a^2}{4c^4}(c^2t'^2 - r'^2)\right]^{-1}, \tag{7.3}$$

and du is given by (3.16) with x^λ replacing z^λ. We also find

$$\tau^2 = \lambda(t', \mathbf{r}')\tau'^2, \tag{7.4}$$

where τ^2 is given by (2.5). In the accelerated system the conform invariant equation (7.1) takes the form

$$\Box'\Phi'(t', \mathbf{r}') = \frac{\kappa}{6}T\Phi'^3(t', \mathbf{r}'), \tag{7.5}$$

where

$$\Phi'(t', \mathbf{r}') = \lambda(t', \mathbf{r}')\Phi(\lambda\tau'^2), \tag{7.6}$$

so that the metric of the accelerated de Sitter background is

$$ds'^2 = \lambda^2(t', \mathbf{r}')\Phi^2(\lambda\tau'^2)\eta_{\mu\nu}\, dx'^\mu\, dx'^\nu. \tag{7.7}$$

The geodesic equation of motion in the de Sitter universe in which mass scintillations are accelerated with acceleration $\mathbf{a}$ reads

$$\frac{d}{du'}\left[\lambda\Phi(\lambda\tau'^2)\frac{dx'^\mu}{du'}\right] = \partial'^\mu[\lambda\Phi(\lambda\tau'^2)]. \tag{7.8}$$

For small accelerations we obtain

$$\Phi'(t', \mathbf{r}') \cong [\lambda\Phi(\lambda\tau'^2)]_{\mathbf{a}=0} + a^n\left[\frac{\partial}{\partial a^n}\lambda\Phi(\lambda\tau'^2)\right]_{\mathbf{a}=0}, \tag{7.9}$$

or

$$\Phi'(t', \mathbf{r}') \cong \Phi(\tau'^2) + \frac{\mathbf{a}\cdot\mathbf{r}'}{c^2}\Phi(\tau'^2). \tag{7.10}$$

The geodesic equation which is

$$\frac{d}{du}\left[\Phi(\tau^2)\frac{dx^\mu}{du}\right] = \partial^\mu\Phi(\tau^2) \tag{7.11}$$

when the mass scintillations of the de Sitter universe are at rest is changed into

$$\frac{d}{du'}\left[\left(1 + \frac{\mathbf{a}\cdot\mathbf{r}'}{c^2}\right)\Phi(\tau'^2)\frac{dx^\mu}{du'}\right] = \partial'^\mu\left[\left(1 + \frac{\mathbf{a}\cdot\mathbf{r}'}{c^2}\right)\Phi(\tau'^2)\right] \tag{7.12}$$

when the mass scintillations are accelerated. Nonrelativistically (7.15) takes the form

$$d^2\mathbf{r}'/dt'^2 \cong -\mathbf{a}. \tag{7.13}$$

Equation (7.13) shows that in the special noninertial frame that corresponds to mass scintillations being accelerated with acceleration $\mathbf{a}$, there is an inertial force proportional to $-\mathbf{a}$ and acting on the test particle. Since the mass scintillations provide the only system of reference we may say that the test particle has acceleration $\mathbf{a}$ with respect to the inertial frame. Viewed from the test particle the universe no longer appears homogeneous and new forces due to the nonhomogeneity appear which we call inertial forces.

We shall presently show that the inertial force in the accelerated system has the right magnitude. If $\bar{m}$ is the gravitational mass of the test particle we have

$$\frac{d}{du'}\left[\bar{m}\Phi'(x')\frac{dx'^\nu}{du'}\right] = \partial'^\nu[\bar{m}\Phi'(x')]. \tag{7.14}$$

As in (6.18) defining the inertial mass by

$$\bar{m}_{\text{in}} = \bar{m}\Phi(x) = \bar{m}\Phi(0)[1 - (\tau^2/4R^2)]^{-1}, \tag{7.15}$$

we see that the inertial mass is a scalar density of weight ¼. The equation of motion in the accelerated system takes the form

$$\frac{d}{du'}\left(\bar{m}'_{\text{in}}\frac{dx'^{\mu}}{du'}\right) = \partial'^{\mu}\bar{m}'_{\text{in}}. \tag{7.16}$$

Now we have

$$\bar{m}'_{\text{in}} = \bar{m}\Phi(0)\lambda(t', \mathbf{r}')\left(1 - \frac{\lambda\tau'^2}{4R^2}\right) \cong \left(1 + \frac{\mathbf{a}\cdot\mathbf{r}'}{c^2}\right)\bar{m}_{\text{in}}, \tag{7.17}$$

so that

$$\frac{d}{du'}\left(\bar{m}'_{\text{in}}\frac{dx'^{\rho}}{du'}\right) \cong \frac{d}{du'}\left(\bar{m}_{\text{in}}\frac{dx'^{\rho}}{du'}\right) \cong -\bar{m}_{\text{in}}a^{\rho}, \tag{7.18}$$

and the inertial force caused by the action on the test particle of the accelerated mass scintillations is shown to be equal to the inertial mass of the test particle times minus its acceleration with respect to the mass scintillations of the de Sitter background.

VIII. COMPARISON WITH OTHER THEORIES

In this section we compare our reinterpretation of Einstein's theory with some of the modifications of Einstein's gravitational equations proposed by certain authors. Hoyle (*12*), Yilmaz (*23*), Brans and Dicke (*14*) have all introduced a scalar field in General Relativity in addition to the fundamental metric tensor $g_{\mu\nu}$. Unlike in Weyl's geometry the scalar fields that occur in these theories have no geometrical meaning.

Instead of Einstein's equations (3.9), Hoyle takes

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu} + (\partial_\mu C)_{;\,\nu}, \tag{8.1}$$

where C is a fundamental scalar field. It serves to give expression to Weyl's postulate in cosmology and hence to define the basic cosmological frame. Hoyle then shows that (8.1) admits the de Sitter metric as a solution if the universal length of $(\partial_\mu C)$ is proportional to the curvature of the universe. We note that, written in the form (3.11) Einstein's equations have a similar structure to (8.1) and, as shown in Section IV, they admit the de Sitter solution.

Yilmaz adds to the right hand side of (3.9) a tensor proportional to the energy momentum tensor of a scalar field ψ, namely

$$t_{\mu\nu} = (\partial_\mu\psi)(\partial_\nu\psi) - \tfrac{1}{2}\delta_{\mu\nu}(\partial_\lambda\psi)(\partial_\lambda\psi). \tag{8.2}$$

Our equations (3.11) contain $t_{\mu\nu}$ as well as additional terms depending on the scalar field. Both these authors claim that the modified equation throws more light on Mach's principle (*23*, *13*).

The theory that the equations (3.11) resemble most is the one proposed by Brans and Dicke with the specific purpose of incorporating Mach's principle in General Relativity. These authors introduce an additional scalar field φ and couple it to the metric tensor $g_{\mu\nu}$ by the following heuristic method: they start from Einstein's variational principle which in our definition (3.8) of $R_{\mu\nu}$ reads

$$\delta \int [R - (16\pi G/c^4)L]\sqrt{-g}\, d^4x = 0, \tag{8.3}$$

where L is the Lagrangian for matter. They divide the integral by the gravitational constant G, then replace G^{-1} by a scalar field φ. For consistency, a term proportional to the Lagrangian of the field φ is then added to the integrand. The variation of the integral thus obtained gives the basic equations of Brans and Dicke which are very similar to (3.11). An undetermined constant ω expresses the strength of the coupling of the scalar field to the metric tensor.

We first remark that equations almost identical to (3.11) would have been obtained by a slight modification of Brans and Dicke's procedure. Instead of dividing (8.3) by G we could have divided by G^2 and then replace G^{-1} by φ, adding a term proportional to the Lagrangian of the scalar field. This modified procedure gives as a new variational principle

$$\delta \int [\varphi^2 R - \varphi(16\pi/c^4)L + \omega(\varphi_{,\nu})(\varphi^{,\nu})]\sqrt{-g}\, d^4x. \tag{8.4}$$

The resulting equations are exactly of the form (3.11) and the two theories become identical if we take $\omega = 6$ and impose the condition

$$\sqrt{-g} = 1. \tag{8.5}$$

To see that the theory thus obtained is the same as Einstein's theory, we do not modify Einstein's variational principle (8.3), but simply re-express the integrand in terms of the tensor density $\gamma_{\mu\nu}$ and the scalar density φ defined by (1.5) and (1.3). From (3.5) we obtain

$$R = \varphi^{-2}\mathfrak{R} - 6\varphi^{-3}\Box_{(\gamma)}\varphi. \tag{8.6}$$

Since we also have

$$\sqrt{-g} = \varphi^4,$$

using a Lagrangian density $\mathfrak{L}$ similar to expression (4.14) for T, we find

$$\delta \int [\varphi^{-2}\mathfrak{R} - 2\kappa\varphi^{-3}\mathfrak{L} - 6\varphi^3\Box_{(\gamma)}\varphi]\varphi^4\, d^4x = 0, \tag{8.7}$$

where κ is the constant that appears in (3.9). Remembering the Machian condition that φ vanishes at infinity, we can rewrite (8.7) in the form

$$\delta \int [\varphi^2 \mathcal{R} - 2\kappa\varphi\mathcal{L} + 6(\varphi_{,\nu})(\varphi^{,\nu})]\, d^4x = 0, \tag{8.8}$$

which is the same as the heuristic Lagrangian in (8.4) with the condition (8.5) and $\omega = 6$, provided we work in the inertial system.

Another remark concerns the equation (5.4). This equation which serves to determine the scalar field φ is truly linear in the conformal approximation, that is, when the space is almost conformally flat. In theories like the one by Brans and Dicke, in which a nongeometrical scalar field is superimposed to the metric, the D'Alembert operator which occurs in an equation such as (5.4) is associated with the metric $g_{\mu\nu}$ so that it is only linear in the weak field approximation when space is almost flat. In our case (5.4) becomes linear even for large $g_{\mu\nu}$ provided the deviations from the conformally flat structure are small. Thus one of the difficulties in Brans and Dicke's theory which is associated with the need to justify the weak field approximation in a cosmological universe does not exist in our scheme based on Einstein's original theory.

It may also be noted that the separation of the metric tensor into a scalar density φ and a tensor density $\gamma_{\mu\nu}$ with unit determinant is analogous to the separation of the four-vector field into its scalar and vector parts in Sciama's model (*10*, *11*). The quantity φ which describes the cosmological structure plays a role similar to that of the time component of Sciama's four-vector field.

From the standpoint of special relativity our separation corresponds to the separation of $g_{\mu\nu}$ into a Minkowski scalar φ describing a spin zero field and a traceless Minkowski tensor $h_{\mu\nu}$ associated with the weak deviations of $\gamma_{\mu\nu}$ from the flat metric describing a spin two field. Expressed in terms of the inertial coordinates, Einstein's theory is cast in a special relativistic form in accordance with the formulations of Gupta (*24*), Feynman (*25*), and Thirring (*17*).

The interpretation of General Relativity proposed here suggests that quantization of General Relativity should lead to two kinds of gravitons, namely, spin zero gravitons associated with cosmological effects and spin two gravitons giving rise to gravitational attraction.

IX. CONCLUDING REMARKS

To emphasize the difference between the Machian solution we have obtained and the standard solution corresponding to the flatness of space at infinity, we rewrite the de Sitter solution of the Einstein equations (3.9) in the weak gravitational field approximation for a point mass m in empty space. We have

$$ds^2 \cong \left(1 - \frac{\kappa c^2}{4\pi}\frac{m}{r}\right) c^2\, dt^2 - \left(\delta_{kl} + \frac{\kappa c^2}{4\pi}\frac{m}{r}\frac{x_k x_l}{r^2}\right) dx^k\, dx^l. \tag{9.1}$$

Here κ is related to the gravitational constant G by the relation,

$$G = c^4\kappa/8\pi, \tag{9.2}$$

so that the sign and magnitude of the gravitational force depend on the value of the constant κ. When space is empty ($m = 0$), we have

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{9.3}$$

that is, the metric of euclidean space.

In our reformulation of Einstein's theory we have a new solution of (39) for the gravitational field of a mass point m immersed in a uniform universe with with total mass M' and radius of curvation R'. According to (5.2), (6.7), any (6.8) an approximate expression for the metric is

$$ds^2 = \varphi^2\gamma_{\mu\nu}\, dx^\mu\, dx^\nu \cong \varphi_0^2\gamma_{\mu\nu}\, dx^\mu\, dx^\nu, \tag{9.4}$$

or

$$ds^2 \cong \left(\frac{\kappa c^2}{24\pi}\frac{M'}{R'}\right)^2 \left[\left(1 - 6\frac{R'}{M'}\frac{m}{r}\right)c^2\, dt^2 - \left(\delta_{kl} + 6\frac{R'}{M'}\frac{m}{r}\frac{x_k x_l}{r^2}\right) dx^k\, dx^l\right]. \tag{9.5}$$

From the form (5.19) of φ_0 one may verify that away from matter ($r \to \infty$), φ_0 vanishes so that the metric tensor tends to zero at infinity, unlike the metric tensor in (9.1) which tends to $\eta_{\mu\nu}$. Furthermore, since the over-all multiplicative constant drops out of the equation of motion, we see that the gravitational constant, unlike in (9.2), does not depend on κ, but is determined by the cosmological structure, having the positive value

$$G = 3c^2R'/M'. \tag{9.6}$$

The constant κ corresponds to a constant scale transformation of the coordinates and cannot be measured by studying the motion of a body in a gravitational field. However, it must be positive to lead to a finite M'. Thus the observed finiteness of G implies a spatially closed universe. This is in agreement with Hönl's results (*4*, *36*).

Turning now to the case of an empty universe, from (9.5) we conclude that

$$\lim_{M'\to 0, m\to 0} ds^2 = 0. \tag{9.7}$$

Thus, if the curvature of the universe is kept fixed as the universe is depleted of matter, the gravitational constant grows and the metric disappears, so that there is no geometry in empty space. We conclude that Machian boundary conditions do not allow empty space solutions of Einstein's equations.

From (6.18) it follows that the inertial mass of a test particle is also determined by the cosmological mass distribution and that it also vanishes in an empty universe.

The foregoing results seem to show that Mach's principle is indeed incorporated in Einstein's equations (3.9). In agreement with Fock's view (*30, 31*), the uniformity of the cosmological background serves to define an inertial coordinate system in which unambiguous boundary conditions can be enunciated. In accordance with Einstein's (*1, 2*) and Wheeler's (*9*) expectations, these boundary conditions lead to solutions which exhibit Mach's principle.

Another result concerns the time dependence of the gravitational constant. This follows from the fact that G^{-1} is proportional to φ, that is, to a power of the determinant of the metric tensor. The possibility of a variable G had already been exploited in the theories of Milne (*21*), Dirac (*22*), Jordan (*18*), and Brans and Dicke. The exact time dependence of G is however determined by the form of the function F in (2.18), which is related to the deviations from the de Sitter background caused by the presence of stable matter in the universe.

More light is also thrown on the Machian interpretation of centrifugal and Coriolis forces by the relation (9.6), since Thirring and Lense (*3*) and others (*2, 7*) have shown that this is the relation which allows the interpretation of inertial forces in a rotating frame as being gravitational forces due to the action of distant matter. This leads to the complete relativity of rotational motion (*36, 37, 38*).

Finally, an important problem that arises from our discussion concerns the stability and uniqueness of the cosmological structure. We have assumed the universe to have a de Sitter geometry in the first approximation. This corresponds, as we have seen, to a background of uniformly spread mass scintillations which could be interpreted as virtual pairs of massive particles in a quantum field theoretical picture. In the second approximation, taking the average contribution of uniformly distributed stable matter into account, we may regard the overall geometry of the universe as conformally flat. Repulsive forces between stable bodies are introduced at this stage. In the third approximation we also allow for the deviations of the Riemannian geometry from the conformally flat structure and thus introduce attractive gravitational forces which are superimposed on an expanding universe. This leads us to question the validity of these successive approximations to the geometry of space-time. Now, in a conformally flat universe we have found the relation (9.6) in which, according to (6.13), part of the total mass M', namely, $N\mu$ comes from stable matter and another part $M(0)$ from the mass scintillations of the de Sitter background. If our approximation is a good one, we must have

$$M(0) \gg N\mu. \qquad (9.8)$$

According to McVittie's (*39*) discussion of recent cosmological data, if we take as M' the stable mass in the universe ($M(0) = 0$), then the equation (9.6) is off by a factor of about 30. This suggests that

$$N\mu/M(0) \cong 3\%, \tag{9.9}$$

and thus our approximation seems to be justified in the present state of the universe. However, it remains to be seen if a universe which has an approximately conformally flat structure and deviates little from a de Sitter space-time will remain so in the course of the cosmical evolution governed by Einstein's equations. Such a study might help us understand whether the remarkable uniformity of our cosmological background is merely accidental or not. The arguments presented in this paper seem to require a cosmological background that is conformally flat, spatially closed, and spatially uniform for reasonable Machian solutions to exist. Uniformity with respect to time is not necessary, though convenient for calculations. If this is the case, then the de Sitter group should have no global validity. A cosmological structure admitting the 6-parameter group of an Einstein space at each instance would be sufficient for the separation of local and cosmological effects implied by Mach's principle.

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