The slice theorem in Kähler geometry

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The Hamiltonian group $\mathcal{G}$ of (M, ω) acts by holomorphic isometries.

Moment map

The scalar curvature is the moment map for $\mathcal{G}$ acting on $\mathcal{AC}_\omega$. (A. Fujiki '92, S. Donaldson '94)

$$\mu : \mathcal{AC}_\omega \to C^\infty_0(M)$$
$$\mu(J) = S - \overline{S}$$

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(M, ω, J_0) cscK with isometry group K.

Finite dimensional slice

There is a ball $B \subset \tilde{H}^1$ in the Kuranishi space and a section, through J_0

$$\Phi : B \to \mathcal{AC}^i_\omega$$

So that μ restricts to

$$\nu : (B, \Omega) \to \mathfrak{k}^*$$

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Theorem

Let $J = \Phi(x)$ for $x \in B$. Then (M, J) admits a cscK metric in the Kähler class $[\omega] \in H^2(M, \mathbb{R})$ if and only if the orbit $K^{\mathbb{C}} \cdot x \subset B$ is polystable.

This is due to G Székelyhidi, T. Brönnle 2010, but with gaps in argument.

Applications

Some applications of the results.

- Study of deformations of constant scalar curvature Kähler metrics, Sasakian metrics, and Higgs bundles (C. Tippler, C. van Coevering, S. Simanca, Y. Fan)
- Proof that small complex deformations of cscK metrics have K-energy bounded below. (V. Tosatti)
- Slice theorem has been used to construct (course) moduli of cscK metrics, as a complex analytic space (R. Dervan, P. Naumann)
- Moduli space of constant scalar curvature Sasakian metrics (C. van Coevering)

Kähler geometry

A *Kähler manifold* is a complex manifold (M, J) with an Hermitian metric g compatible with the complex structure:

g is Kähler if

- Almost Hermitian: $g(J\cdot, J\cdot) = g(\cdot, \cdot)$.
- J parallel: $\nabla J = 0$, where ∇ Levi-Civita connection.

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An Hermitian metric g on a complex manifold (M, J) is Kähler if and only if $\omega(\cdot, \cdot) := g(J\cdot, \cdot)$ is closed, $d\omega = 0$.
The Kähler form ω is a type $(1, 1)$ symplectic form.
The Kähler manifold can be denoted (M, J, ω) with $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$.
If J is merely an almost complex structure then (M, J, ω), $d\omega = 0$, is an *almost Kähler manifold*.

Examples

Examples are abundant.

- Any algebraic manifold $M \subset \mathbb{CP}^N$. The Fubini Study metric on $\mathbb{CP}^N$ restricts to a Kähler metric.
- A complex torus $M = \mathbb{C}^n/\Lambda$, Λ is a lattice of rank 2n, has the flat Kähler metric.
- Not all examples are algebraic: A compact complex manifold M is algebraic if and only if it admits a Kähler form ω with $[\omega] \in H^2(M, \mathbb{Z})$. Then $\omega \in c_1(L)$ for an ample holomorphic line bundle L.

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Kodaira embedding

Let L be an ample line bundle on M, then

$$\iota_{L^r} : M \to \mathbb{P}\Big(H^0(M, \mathcal{O}(L^r))^*\Big)$$

is an embedding for $r >> 1$.

- In the algebraic case M is "polarized" by L, (M, L). But in general $[\omega] \in H^2(M, \mathbb{R})$ is an irrational class.

Curvature

$\partial\overline{\partial}$-lemma

If $\omega_1 \in [\omega_0]$ another Kähler form in the same cohomology class then

$$\omega_1 - \omega_0 = \sqrt{-1}\partial\overline{\partial}f, \quad \text{for some } f \in C^\infty(M).$$

Kähler metrics on (M, J) in a fixed Kähler class $[\omega] \in H^2(M, \mathbb{R})$ are parametrized by potential functions $f \in C^\infty(M)$.

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Ricci curvature

The Ricci curvature of the metric g has a simple expression on Kähler manifolds:

- The Ricci form $\rho(\cdot,\cdot) := \mathrm{Ric}(J\cdot,\cdot)$ is the $(1,1)$ form associated to Ric.
- $\rho = \sqrt{-1}\overline{\partial}\partial \log\left(\det \omega_{\alpha\overline{\beta}}\right)$, where in local coordinates $\omega = \sqrt{-1}\sum \omega_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}}$.

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$$\mathrm{Ric} = \lambda g$$

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Kahler-Einstein problem

Let $K_M = \Lambda^{n,0} TM$ be the canonical bundle.

- $(\lambda < 0)$ K_M is ample, $[\omega] = \frac{2\pi}{\lambda} c_1(M)$. (Solved S.-T. Yau, T. Aubin, 1976)
- $(\lambda = 0)$ $c_1(M) = 0$. Existence follows from Yau's solution to the Calabi conjecture. (Solved by S.-T. Yau, 1977)
- $(\lambda > 0)$ K_M^{-1} is ample, so (M, J) is Fano. Existence of of K-E metric proved to by equivalent to K-polystability of (M, K_M^{-1}) (X. Chen, S. Donaldson, S. Sun 2012, also G. Tian 2012)

cscK

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A *constant scalar curvature Kähler* metric (cscK) satisfies

$$S = \overline{S}$$

where

$$\begin{aligned}\overline{S} &= \frac{1}{\operatorname{Vol}(M)} \int \frac{1}{(n-1)!} \rho \wedge \omega^{n-1} \\ &= \frac{n[\rho] \cdot [\omega]^{n-1}}{[\omega]^n}\end{aligned}$$

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Yau-Tian-Donaldson Conjecture

A polarized complex manifold (M, L) should admit a cscK metric in the class $c_1(L)$ if and only if (M, L) is K-polystable.

cscK

- One can extend the conjecture to consider Kähler manifolds (M, α) polarized with a possibly irrational Kähler class $\alpha \in H^2(M, \mathbb{R})$. (Z. Sjöström Dyrefelt 2017)
- The Käher-Einstein case was proved by X. Chen, S. Donaldson, S. Sun 2012, see also G. Tian 2012.
- The "only if" part has been proved. (J. Stoppa '08; R. Berman '13; R. Berman, T. Darvas, C. Lu '16)
- For the rest of the conjecture it has been proved that existence of cscK is equivalent to an analytic stability condition, convexity of a Kempf-Ness functional, K-energy. (X. Chen, J. Cheng '18)

Space of metrics

Fix a compact symplectic manifold (M, ω). $\mathcal{AC}_\omega$ is the space of almost complex structures J compatible with ω:

$$\omega(JX, JY) = \omega(X, Y)$$
$$\omega(X, JX) > 0, \text{ for } X \neq 0$$

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$\mathcal{AC}_\omega$ infinite dimensional Kähler manifold

- The *tangent space* $T_J\mathcal{AC}_\omega$ consists of $A \in \mathrm{End}(TM)$ with

$$JA = -AJ \quad \text{and } \omega(AX, Y) + \omega(X, AY) = 0$$

- $\mathcal{AC}_\omega$ has a complex structure $\mathcal{J}$:

$$\mathcal{J}A = J \circ A$$

Space of metrics

- Fix $J_0 \in \mathcal{AC}_\omega$. Let $\mathrm{End}(TM, J_0)_S$ be all tensors $\mu \in \mathrm{End}(TM)$
  1. $J_0 \circ \mu = -\mu \circ J_0$,
  2. μ is symmetric with respect to g_{J_0},
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  Then

$$\mathrm{End}(TM, J_0)_S \to \mathcal{AC}_\omega$$
$$\mu \mapsto J_0 \big(1 - \mu\big) \big(1 + \mu\big)^{-1}$$

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  is a complex analytic chart.
- The Kähler metric is given by

$$G(A, B) = \frac{1}{2} \int \mathrm{tr}\big(AB\big) \frac{\omega^n}{n!}$$

  with the Kähler form

$$\Omega(A, B) = \frac{1}{2} \int \mathrm{tr}\big(JAB\big) \frac{\omega^n}{n!}$$

Moment map

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Theorem (A. Fukiki '92, S. Donaldson '94)

The action of $\mathcal{G}$ on $\mathcal{AC}_\omega$ is hamiltonian with moment map

$$\mu : \mathcal{AC}_\omega \to C_0^\infty(M)$$
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S_J is the scalar curvature of the Chern connection, the usual scalar curvature when J integrable.
The Lie algebra of $\mathcal{G}$ is $C_0^\infty(M)$, hamiltonian functions.

Moment map

For $H \in C^\infty(M)$ the hamiltonian vector field X_H is defined by

$$dH = X_H \lrcorner \omega$$

The infinitesimal action of $C_0^\infty(M)$

$$P : C_0^\infty(M) \to T_J \mathcal{AC}_\omega$$

$$P(H) = \mathcal{L}_{X_H} J$$

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μ being a moment map means

$$\langle Q(\alpha), H\rangle_{L^2} = \Omega\big(\alpha, P(H)\big)$$

Kähler classes

The hamiltonian group $\mathcal{G}$ has no complexification, but we can still describe the orbits of $\mathcal{G}^{\mathbb{C}}$ on integrable $J \in \mathcal{AC}^i_\omega$. Extend

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J_0, J_1 are in the same "orbit" of $\mathcal{G}^{\mathbb{C}}$ if there is a path $\phi_t \in C^\infty_0(M, \mathbb{C})$ and $J_t \in \mathcal{AC}^i_\omega$ joining J_0, J_1

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There is an $f \in \mathrm{Diff}(M)$ with $f^* J_1 = J_0$ and

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The orbit of $\mathcal{G}^{\mathbb{C}}$ is essentially the Kähler class of $([\omega], J_0)$

$$\left\{ \omega + \sqrt{-1}\partial\overline{\partial}\phi \mid \phi \in C^\infty(M),\ \omega + \sqrt{-1}\partial\overline{\partial}\phi > 0 \right\}$$

Deformation complex

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We also have

$$\overline{\partial} : T_{J_0}\mathcal{AC}^i_\omega \to \Omega^{0,2}(T^{1,0})$$

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$$\Lambda^{0,2} \otimes T^{1,0} \cong \Lambda^{0,2} \otimes \Lambda^{0,1} \to \Lambda^3$$

Define $\mathcal{B}^2 = \Gamma(E^2)$.

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We have the elliptic complex

$$C_0^\infty(M, \mathbb{C}) \xrightarrow{P} T_{J_0}\mathcal{AC}^i_\omega \xrightarrow{\overline{\partial}} B^2 \to \cdots$$

and

$$\tilde{H}^1 = \left\{ \alpha \in T_{J_0}\mathcal{AC}^i_\omega | P^*\alpha = \overline{\partial}\alpha = 0 \right\}$$

Slice Theorem

Theorem

There is a ball $B \subset \tilde{H}^1$ around the origin and a K-equivariant map

$$\Phi : B \to \mathcal{AC}_\omega$$

such that the $\mathcal{G}$ orbit of every integrable J near J_0 intersects the image of Φ. If x and x' are in the same $K^{\mathbb{C}}$ orbit, $\Phi(x)$ integrable, then $\Phi(x), \Phi(x')$ are in same $\mathcal{G}^{\mathbb{C}}$ orbit. And moment map restricts to $\mu : \mathcal{AC}_\omega \to C_0^\infty(M)$

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We must perturb Φ_1 to cancel out the $\mathfrak{k}^\perp$ portion of μ

$$\mathfrak{k} \oplus \mathfrak{k}^\perp = C_0^\infty(M).$$

Slice Theorem

Let $U \subset C_0^\infty(M)$ be a neighborhood of 0 so that if $\phi \in U$

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And

$$f_1^*\big(J, \omega + dd^c\phi\big) = \big(F(\phi, J), \omega\big)$$

Note that if $\phi \in L^2_k$ then $F(\phi, J) \in L^2_{k-3}$.

Slice Theorem

Let $V \subset \mathfrak{k}^\perp$ be a small ball. Define map

$$\begin{array}{rccc} G: & B_1 \times V & \to & \mathfrak{k}^\perp \\ & (x, \phi) & \mapsto & \Pi S(F(\phi, \Phi_1(x)) \end{array}$$

where $\Pi : C_0^\infty \to \mathfrak{k}^\perp$ is the projection.

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Differential at origin

$$DG_{(0,0)} : \mathfrak{k}^\perp \to \mathfrak{k}^\perp$$
$$DG_{(0,0)}\psi = P^* P(\psi)$$

is the restriction of a 4-th order elliptic operator.

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Differential at origin

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$$DG_{(0,0)}\psi = P^* P(\psi)$$

is the restriction of a 4-th order elliptic operator.
But if $\phi \in L_k^2$ then $G(x, \phi) \in L_{k-5}^2$, so there is no hope of using the Banach implicit function theorem.

Nash-Moser theorem

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(R. Hamilton, Bull. A.M.S. Vol. 7, No. 1, 1982)

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We note that G is a smooth map of Fréchet spaces. Further, it is a smooth *tame* map. The Nash-Moser theorem is applicable if

$$DG_{(x,\phi)} : \mathfrak{k}^{\perp} \to \mathfrak{k}^{\perp}$$

has a smooth tame inverse for (x, ϕ) in a neighborhood of $(0, 0)$. This follows from the fact that $DG_{(0,0)}$ is elliptic, Fredholm theory, and results in (Hamilton '82).

Finite dimensional GIT

Proposition

After possibly shrinking B, suppose $v \in B$ is polystable for the $K^{\mathbb{C}}$ action on $\tilde{H}^1$. Then there is $v_0 \in B$ in the $K^{\mathbb{C}}$-orbit of v with $\mu(v_0) = 0$.

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Let

$$\nu : \tilde{H}^1 \to \mathfrak{k}$$

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Let

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be the moment map for the flat Kähler structure (Ω_0, J) on $\tilde{H}^1$. The Taylor series gives

$$\mu(tv) = \mu(0) + td\mu_0(v) + \frac{t^2}{2}\frac{d^2}{dt^2}\Big|_{t=0}\mu(tv) + O(t^3)$$

We have $\mu = 0$ and $d\mu_0 = 0$.

Finite dimensional GIT

It is straight forward that

$$\frac{d^2}{dt^2}\mu(tx)\Big|_{t=0} = 2\nu(x).$$

So

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One shows that if x is polystable iff $\|\nu(x)\|$ has a minimum on $K^{\mathbb{C}}$-orbit. Then after possibly shrinking B, $\|\mu(x)\|$ has a minimum x' which must have $\mu(x') = 0$.

K-polystability

K-polystability is defined in terms of 1-parameter degenerations, test configurations. A *test configuration* for (M, L) is a flat $\mathbb{C}^*$-equivariant family

$$\pi : \chi \to \mathbb{C}$$

with $\mathcal{L}$ a relatively ample bundle on χ and $(\chi_t, \mathcal{L}_t) \cong (M, L^r)$ for $t \neq 0$.

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Definition

(M, L) *is K-polystable if* $\mathrm{DF}(\chi, \mathcal{L}) \geq 0$ *for all test configurations. And if* $\mathrm{DF}(\chi, \mathcal{L}) = 0$ *for a test configuration with normal total space,* $\chi \cong M \times \mathbb{C}$*.*

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Definition

(M, L) is K-polystable if $\mathrm{DF}(\chi, \mathcal{L}) \geq 0$ for all test configurations. And if $\mathrm{DF}(\chi, \mathcal{L}) = 0$ for a test configuration with normal total space, $\chi \cong M \times \mathbb{C}$.

Theorem

Let (M, J_0, ω) be cscK. Let $J \in \mathcal{AC}^i_\omega$ be a nearby Kähler structure. If (M, J) is K-polystable then it admits a cscK metric $(M, J, \omega + dd^c\phi)$.

K-polystability

If (M, J) is K-polystable then its $\mathcal{G}^{\mathbb{C}}$ orbit must intersect the slice in a polystable orbit.
Otherwise, using the finite dimensional GIT picture, one constructs a (smooth) degeneration

$$\pi : \chi \to \mathbb{C}$$

where the central fiber χ_0 admits a cscK metric. Thus $\mathrm{DF}(\chi, \mathcal{L}) = 0$, but χ is not a product.

Thank You