Lecture 1: G_2-Geometry and Associative 3-folds

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6th Geometry-Topology Summer School

Overview

Mon: Expository talk.

- Lecture 1 (Parts I, II): Holonomy & G_2-Geometry

Tue: Expository talk.

- Lecture 1 (Part III): Associatives & Coassociatives
- Lecture 2: Nearly-Kähler Geometry & Holomorphic Curves

Wed: Research talk.

- Lecture 3: Closed Holomorphic Curves in $\mathbb{S}^6$

Fri: Research talk.

- Lecture 4: Free-Boundary Holomorphic Curves in NK 6-Mflds

Outline

I. Holonomy

- Definition & Properties
- Berger's List

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II. G_2 Geometry

- The Octonions
- G_2-Structures on 7-Manifolds
- G_2 Holonomy on 7-Manifolds
- Examples

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III. Special Submanifolds

- Calibrations
- Associative 3-folds
- Coassociative 4-folds

Holonomy: Definition

Let (M^n, g) connected Riemannian manifold.
Let $\gamma\colon [0,1] \to M$ loop at $x \in M$. Consider:

$$\begin{aligned} P_\gamma \colon T_xM &\to T_xM \\ P_\gamma(v) &= \text{parallel translation of } v \text{ around } \gamma \end{aligned}$$

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Holonomy group of (M, g) at $x \in M$:

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Group structure is via loop concatenation (just like for $\pi_1(M, x)$):

$$P_\alpha \circ P_\beta = P_{\alpha\beta} \qquad\qquad (P_\alpha)^{-1} = P_{\overline{\alpha}}$$

where $\overline{\alpha}$ is the reverse path of α.

Holonomy: Properties

Parallel translation is an isometry. So:

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Basepoint Independence. Let γ path from x to y. Then

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Properties:

- $\mathrm{Hol}^0(g)$ is connected.
- $\mathrm{Hol}^0(g) \trianglelefteq \mathrm{Hol}(g)$. ($M$ simply-connected $\implies$ $\mathrm{Hol}^0(g) = \mathrm{Hol}(g)$.)
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$\therefore$ $\mathrm{Hol}^0(g)$ is a compact connected Lie group.

Holonomy Algebra; Holonomy Representation

$\mathrm{Hol}^0(g)|_x \leq \mathrm{SO}(T_xM)$ is a compact connected Lie group. Call its Lie algebra

$$\mathfrak{hol}(g)|_x \subset \mathfrak{so}(T_xM) \cong \Lambda^2(T_x^*M)$$

the **holonomy algebra** of (M, g).

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Remark: $\mathrm{Hol}^0(g)|_x$ is not just an abstract group: It comes with an embedding into $\mathrm{SO}(T_xM)$ (i.e.: There is an obvious action on T_xM).

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Question: Why is the holonomy of (M, g) important? Answer: Holonomy is intimately related to:

- Curvature
- Parallel tensor fields
- Extra geometric structure (compatible with g)

Holonomy & Curvature

Let $T = T_x M$. The Riemann curvature tensor $\mathrm{Rm}(g) \in (T^*)^{\otimes 4}$ at $x \in M$ has various symmetries (e.g.: $R_{ijk\ell} = -R_{jik\ell} = -R_{ij\ell k}$):

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Ambrose-Singer Theorem ('53): The holonomy algebra is generated by the curvature:

The holonomy algebra $\mathfrak{hol}(g)|_x \subset \mathfrak{so}(T_xM)$ is the subspace spanned by

$$\left\{ P_\gamma^{-1} \circ R(X,Y)|_y \circ P_\gamma \,\middle|\, \gamma \text{ path from } x \text{ to } y,\ X, Y \in T_yM \right\}$$

where $R(X,Y)|_y \in \mathfrak{so}(T_yM)$ is the Riemann curvature endomorphism.

Holonomy & Parallel Tensors

The holonomy representation determines which tensor fields S are parallel (i.e.: $\nabla S = 0$):

Holonomy Principle: Let $x \in M$.

(a) If S parallel tensor field, then $S|_x$ fixed by the $\text{Hol}(g)|_x$-action on T_xM.

(b) If S_0 fixed by the $\text{Hol}(g)|_x$-action on T_xM, then there is a unique parallel tensor field S with $S|_x = S_0$.

Holonomy & Extra Geometric Structure

Let (M^n, g) Riemannian n-manifold.

Theorem: Let $G \leq \mathrm{O}(n)$. The following are equivalent:

(i) $\mathrm{Hol}(g) \leq G$.

(ii) There exists a G-structure on M that is g-compatible and torsion-free.

We won't define the terms "G-structure" or "g-compatible" or "torsion-free" here.

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A "torsion-free $\mathrm{U}(\frac{n}{2})$-structure" is just a "Kähler structure": A pair (g, J), where J is a g-parallel, g-orthogonal complex structure. Therefore:

$$\mathrm{Hol}(g) \leq \mathrm{U}(\tfrac{n}{2}) \iff g \text{ is a Kähler metric.}$$

Holonomy: Classification (Products)

Products: If $g \cong g_1 \times g_2$, then $\text{Hol}(g) = \text{Hol}(g_1) \times \text{Hol}(g_2)$.

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Def: Let (M, g) Riemannian. Say:

- g **globally reducible** if: $g \cong g_1 \times g_2$.
- g **locally reducible** if: Every point in M has globally reducible neighborhood.
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de Rham Decomposition: Suppose $\text{Hol}(g)|_x$-representation is reducible at some $x \in M$. Then:

- (Local) $\text{Hol}^0(g)$ is a product group and g locally reducible.
- (Global) If also (M, g) is complete and simply connected, then $\text{Hol}(g)$ is a product *of holonomy* groups and g globally reducible.

Holonomy: Classification (Locally Symmetric Spaces)

Say (M, g) is **locally symmetric** if:

$$\nabla R = 0.$$

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Summary: If g locally reducible, or g locally symmetric, then we understand $\text{Hol}^0(g)$.

What about the other cases (g not locally reducible and g not locally symmetric)?

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$\text{SU}(\frac{n}{2})$	$n = 2m$		
$\text{Sp}(\frac{n}{4})$	$n = 4m$		
$\text{Sp}(\frac{n}{4})\text{Sp}(1)$	$n = 4m$		
G_2	$n = 7$		
$\text{Spin}(7)$	$n = 8$		
$\text{Spin}(9)$	$n = 16$		

Holonomy: Berger's List

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Alekseevsky ('68), Brown-Gray ('72): Hol $=$ Spin(9) $\implies$ Symmetric

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Open: Are there compact, Ricci-flat, irreducible (M^5, g)?

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Fact: Let A be a normed division algebra. Define

$$\mathrm{Aut}(A) := \{g \in \mathrm{GL}(A) \colon g(xy) = g(x)g(y),\ \forall x, y \in A\}.$$

Then:

$$\begin{aligned} \mathrm{Aut}(\mathbb{R}) &= \{\mathrm{Id}\} \\ \mathrm{Aut}(\mathbb{C}) &\cong \mathbb{Z}_2 \\ \mathrm{Aut}(\mathbb{H}) &\cong \mathrm{SO}(3) \\ \mathrm{Aut}(\mathbb{O}) &\cong \mathrm{G}_2 \end{aligned}$$

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One can take $\mathrm{G}_2 = \mathrm{Aut}(\mathbb{O})$ as the **definition** of G_2.

$\therefore$ To understand G_2, we need to understand $\mathbb{O}$.

Normed Division Algebras

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- A is a finite-dim $\mathbb{R}$-algebra (not necessarily associative) with unit 1;
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(Note: Generalizing "normed division algebra" to allow $\langle\cdot,\cdot\rangle$ non-deg. (not just positive-definite), one gets three more algebras: $\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{H}}$, $\widetilde{\mathbb{O}}$.)

Normed Division Algebras

A **normed division algebra** is a pair $(A, \langle \cdot, \cdot \rangle)$ where:

- A is a finite-dim $\mathbb{R}$-algebra (not necessarily associative) with unit 1;
- $\langle \cdot, \cdot \rangle$ is a positive-definite inner product whose norm $\| \cdot \|$ satisfies

$$\|xy\| = \|x\|\|y\|, \quad \forall x, y \in A.$$

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(Note: Generalizing "normed division algebra" to allow $\langle \cdot, \cdot \rangle$ non-deg. (not just positive-definite), one gets three more algebras: $\widetilde{\mathbb{C}}$, $\widetilde{\mathbb{H}}$, $\widetilde{\mathbb{O}}$.)

Def (unhelpful): The **octonions** $\mathbb{O}$ are the normed division algebra that isn't $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

The Octonions

Def ($8 = 4 + 4$): The **octonions** $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$ are pairs of quaternions with multiplication law

$$(a, b) \cdot (c, d) := (ac - \overline{d}b, da + b\overline{c}).$$

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Def ($8 = 1+7$): The **octonions** $\mathbb{O} = \text{span}_{\mathbb{R}}(1, e_1, \ldots, e_7)$ are objects of the form

$$x = x_0 + x_1 e_1 + \cdots + x_7 e_7$$

with multiplication defined by the table:

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	$-e_3$	-1	e_1	e_6	$-e_7$	$-e_4$	e_5
e_3	e_2	$-e_1$	-1	$-e_7$	$-e_6$	e_5	e_4
e_4	$-e_5$	$-e_6$	e_7	-1	e_1	e_2	$-e_3$
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e_6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	-1	e_1
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Algebra with $\mathbb{O}$

Let $\mathrm{Re}(\mathbb{O}) := \mathrm{span}(1) \simeq \mathbb{R}$ and $\mathrm{Im}(\mathbb{O}) := \mathrm{span}(e_1, \ldots, e_7) \simeq \mathbb{R}^7$, so

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Writing $x \in \mathbb{O}$ as $x = \mathrm{Re}(x) + \mathrm{Im}(x)$, define

$$\overline{x} := \mathrm{Re}(x) - \mathrm{Im}(x).$$

Note: $\mathrm{Re}(x) = \frac{1}{2}(x + \overline{x})$ and $\mathrm{Im}(x) = \frac{1}{2}(x - \overline{x})$. Careful:

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Identities: Let $x, y, z \in \mathbb{O}$. Then:

$$\langle xz, yz \rangle = \langle x, y \rangle \|z\|^2 = \langle zx, zy \rangle$$
$$\langle x, y \rangle = \mathrm{Re}(x\overline{y}) = \mathrm{Re}(\overline{x}y).$$

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The octonions are **not** associative, but they are **alternative**:

$$x(xy) = x^2 y \qquad (xy)y = xy^2 \qquad (xy)x = x(yx).$$

Geometry of $\mathrm{Im}(\mathbb{O}) = \mathbb{R}^7$

$\mathrm{Im}(\mathbb{O}) \simeq \mathbb{R}^7$ carries special geometric structures:

- The **vector cross product** $\times \colon \Lambda^2(\mathbb{R}^7) \to \mathbb{R}^7$:

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Analogy: In $\mathbb{C}^n = \mathbb{R}^{2n}$, the **Kähler** 2**-form** is $\omega(x, y) := \langle Jx, y \rangle$.
The pair (J, ω) on $\mathbb{R}^{2n}$ is analogous to $(\times, \phi_0)$ on $\mathbb{R}^7$.

$\mathrm{Im}(\mathbb{O})$ and G_2

Let $V = \mathbb{R}^7$. For each 3-form $\gamma \in \Lambda^3(V^*)$, define a symmetric bilinear form

$$\begin{aligned} B_\gamma \colon \mathrm{Sym}^2(V) &\to \Lambda^7(V^*) \\ B_\gamma(x,y) &:= \tfrac{1}{6}\,(\iota_x\gamma) \wedge (\iota_y\gamma) \wedge \gamma. \end{aligned}$$

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Upshot: If γ is definite, then

$$\tfrac{1}{6}\,(\iota_x\gamma) \wedge (\iota_y\gamma) \wedge \gamma = g_\gamma(x, y)\,\mathrm{vol}_\gamma$$

for some positive-definite inner product g_γ and orientation form vol_γ on $V = \mathbb{R}^7$.

Consider the usual $\mathrm{GL}_7(\mathbb{R})$-action on $V = \mathbb{R}^7$. This gives a $\mathrm{GL}_7(\mathbb{R})$-action on $\Lambda^3(V^*)$ via change-of-coordinates (pullback):

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Fundamental Facts: Let $\phi_0 \in \Lambda^3(V^*)$ the associative 3-form.

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Corollary: $\mathrm{Orbit}(\phi_0) = \Lambda^3_+(V^*)$ is an **open set** in $\Lambda^3(V^*) \simeq \mathbb{R}^{35}$.

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Summary: A G_2-structure $\varphi \in \Omega^3(M)$ identifies

$$(T_xM, \varphi|_x) \simeq (\mathrm{Im}(\mathbb{O}), \phi_0).$$

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Warnings:

• The map $\varphi \mapsto g_\varphi$ is not injective. Different G_2-structures may induce the same metric.

• The map $\varphi \mapsto g_\varphi$ is highly nonlinear.

• The Hodge star $*$ is determined by $(g_\varphi, \mathsf{vol}_\varphi)$, which depends nonlinearly on φ.

Classes of G_2-Structures

Let φ be a G_2-structure.

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Joyce Criterion ('96): Let (M^7, g) compact, $\text{Hol}(g) \leq G_2$. Then:

$$\text{Hol}(g) = G_2 \iff \pi_1(M) \text{ finite.}$$

G_2 Holonomy

Strategy: Let M^7 orientable, spin. To construct g with $\mathrm{Hol}(g) = G_2$:

1. (Non-trivial) Construct torsion-free G_2-structure φ. i.e.: Find G_2-structure solving the nonlinear PDE system

$$\begin{aligned} d\varphi &= 0 \\ d * \varphi &= 0. \end{aligned}$$

Then $\mathrm{Hol}(g_\varphi) \leq G_2$.

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Remark: If M^7 compact, then there are topological obstructions to existence of G_2-holonomy metrics. Necessary conditions are:

- $\pi_1(M)$ finite
- $b^3(M) \geq 1$ (where $b^3(M) = \dim(H^3(M; \mathbb{R}))$)
- $p_1(M) \neq 0$ (where $p_1(M) =$ first Pontryagin class).

Hard Open Problem: Let M^7 compact (orientable, spin). Find necessary and sufficient conditions for M^7 to admit G_2-holonomy metrics.

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Constructing torsion-free G_2-structures is not easy. Most (all?) examples use one of the following ideas:

Idea 1: $7 = 6 + 1$. (Build M^7 from Calabi-Yau X^6 or nearly-Kähler X^6)

Idea 2: $7 = 4 + 3$.

Local Examples

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Bryant ('87): Constructed the first explicit example of $\mathsf{Hol}(g) = \mathsf{G}_2$.
Consider $F^6 = \mathsf{SU}(3)/T^2$. Equip F with a certain homogeneous metric g_F (its homogeneous nearly-Kähler metric). Then

$$M^7 := \mathsf{Cone}(F^6) = \mathbb{R}^+ \times \frac{\mathsf{SU}(3)}{T^2}$$

with cone metric

$$g_M := dr^2 + r^2 g_F$$

has $\mathsf{Hol}(g_M) = \mathsf{G}_2$.

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As smooth manifolds: Their examples are total spaces of vector bundles:

$$\pi\colon (M^7, g_M) \to (B, g_B).$$

- $\Lambda^2_+(\mathbb{S}^4)$ is a rank 3 vector bundle over $\mathbb{S}^4$
- $\Lambda^2_+(\mathbb{CP}^2)$ is a rank 3 vector bundle over $\mathbb{CP}^2$
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Geometrically:

- Bundle metrics: Metric has form

$$g_M = f(r)^2\alpha + g(r)^2\,\pi^*(g_B)$$

where $\alpha|_{\text{fiber}} =$ flat metric and $r =$ radius in fibers.

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- At infinity: They are all asymptotic to G_2-holonomy cones.

Compact Examples: Joyce's Theorem

Idea: Let (M^7, φ) compact with **closed** G_2-structure that is "almost" co-closed:

$$d\varphi = 0 \quad \text{and} \quad d*\varphi \text{ small.}$$

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Joyce's Perturbation Theorem ('96): Fix constants $A_1, A_2, A_3 > 0$. Then there exists an interval $(0, T]$ with $T > 0$ and a constant $K > 0$ such that:

If M^7 compact has a one-parameter family $(\varphi_t, \psi_t)_{t \in (0,T]}$ of almost G_2-structures satisfying

(a) $\|\psi_t\|_{L^2} \leq A_1 t^4$ and $\|\psi_t\|_{C^0} \leq A_1 t^{1/2}$ and $\|d^*\psi_t\|_{L^{14}} \leq A_1$
(b) $\mathrm{inj}(g_{\varphi_t}) \geq A_2 t$
(c) $\|R(g_{\varphi_t})\|_{C^0} \leq A_3 t^{-2}$

then M^7 admits **torsion-free** G_2-structures $\widetilde{\varphi}_t$ with

$$\|\widetilde{\varphi}_t - \varphi_t\|_{C^0} \leq K t^{1/2}.$$

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Open Questions:

- If M^7 compact admits closed G_2-structure, must it also admit a torsion-free G_2-structure?
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- How can we distinguish G_2-holonomy 7-manifolds?

Outline

I. Holonomy

- Definition & Properties
- Berger's List

II. G_2 Geometry

- The Octonions
- G_2-Structures on 7-Manifolds
- G_2 Holonomy on 7-Manifolds
- Examples

III. Special Submanifolds

- Calibrations
- Associative 3-folds
- Coassociative 4-folds