Constant Mean Curvature surfaces and integrable systems

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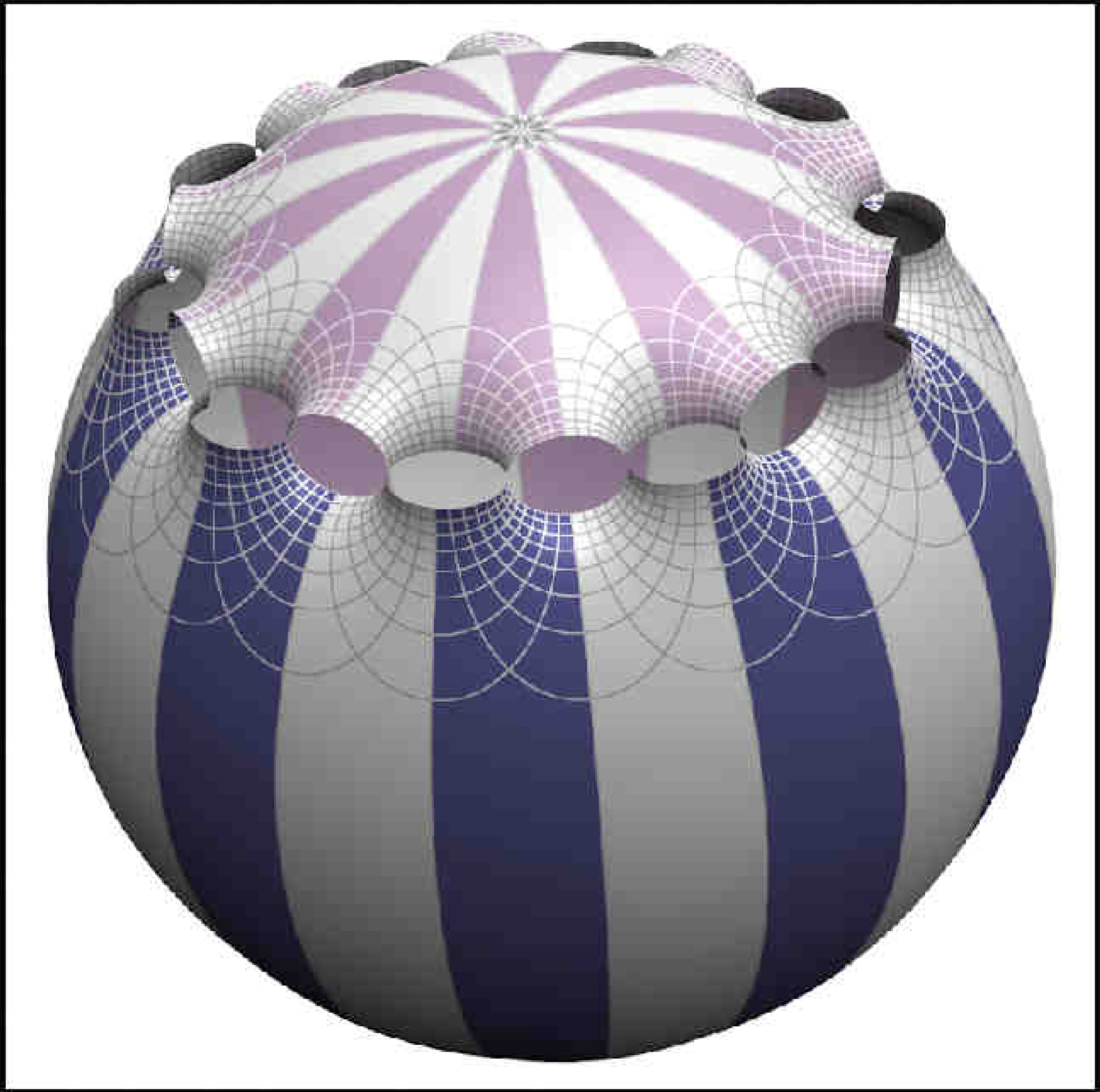

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Surfaces in $\mathbb{R}^3$

$\underline{\Sigma}$ is a compact Riem. surface

i.e. complex str. $J : TM \to TM$

$J^2 = -id$

$\rightsquigarrow$ atlas $z = x + iy$

$i\,dz = dz \circ J$

$f : \Sigma \to \mathbb{R}^3$ <u>conformal</u> immersion

- $df \circ J = N \times df$ N oriented unit normal

Identifications: $\mathbb{R}^3 \cong su(2)$

$\underline{\langle A, B \rangle} = \frac{1}{2}\, tr\,(A \bar{B}^t)$

quadratic form is Det

unit normal $N : \Sigma \to su(2)$

$\det(N) = \underline{1}$

$$W := \underline{su(2)} := \Sigma \times su(2) = T\Sigma \oplus \mathcal{N}$$

$$\mathcal{N} = \mathbb{R} N$$

Observation: $Ad(N)(A) = N^{-1} A N$

$$Ad(N)_{|T\Sigma} = -1 \qquad Ad(N)_{|\mathcal{N}} = +1$$

$$d : \Gamma(W) \to \Omega^1(\Sigma, W)$$

$$W = T\Sigma \oplus N \quad \rightsquigarrow \quad d = \nabla + \mathbb{I}$$

diagonal $\qquad$ off-diagonal

$\Leftrightarrow$ ∇ commute with $Ad(N)$

$\mathbb{I}$ anti-commutes with $Ad(N)$

$$(Ad(N))^2 = id \quad \rightsquigarrow \quad \nabla = \frac{1}{2}\left(d + d \cdot Ad(N)\right)$$

$$\mathbb{I} = \frac{1}{2}\left(d - \underline{d \cdot Ad(N)}\right)$$

Claim: $\nabla(A) = \frac{1}{2}[N dN, A]$ $\qquad A \in \Gamma(\Sigma, \omega)$

computation: $\nabla(A) = \frac{1}{2}\left(d - Ad(N^{-1})\, d\, Ad(N)\right) A$

$= \frac{1}{2} dA - \frac{1}{2} N\, d(N^{-1} A N)\, N^{-1}$

$= \ldots = \frac{1}{2}[N dN, A]$.

Spin: Consider $V = \underline{\mathbb{C}}^2 = \Sigma \times \mathbb{C}^2$ (r.)

$$\underline{su(V) \cong W} = T\Sigma \oplus N$$

N unit normal $\det(N) = \underline{1}$

N has ev $\pm i$

$$\underline{V = L \oplus L^\perp}$$

$\pm i$ eigenline bundles

decompose $d = \nabla + \Phi$ ∇ is diagonal

$\Phi = \frac{1}{2} N dN$ $\nabla = d - \frac{1}{2} N dN$ Φ is off-diag.

$\oplus$ $V = L \oplus L^{-1}$ $\rightsquigarrow$ $\mathrm{End}_0(V) = \mathbb{C} \otimes \mathfrak{su}(V)$

$L^{-1} \cong L^*$

$$\cong \underline{\mathbb{C}} N \oplus L^2 \oplus L^{-2}$$

$$\cong N \otimes \mathbb{C} \oplus T\Sigma \otimes \mathbb{C}$$

$$T\Sigma \otimes \mathbb{C} = T^{1,0}\Sigma \oplus T^{0,1}\Sigma$$

$$J = \quad i \qquad -i$$

$$\begin{matrix} z = x + iy & \frac{\partial}{\partial z} & \frac{\partial}{\partial \bar{z}} \end{matrix}$$

$L^2 \cong T^{1,0}\Sigma$ $\qquad$ $L^{-2} \cong T^{0,1}\Sigma$

$$K = \{ \omega : TM \to \mathbb{C} \mid \omega \circ J = * \omega \overset{!}{=} i \omega \}$$

$$\bar{K} = \{ \eta : TM \to \mathbb{C} \mid \omega \circ J = * \omega = - i \omega \}$$

$$T^{1,0} \Sigma \cong K^{-1}$$

- $(L^*)^2 = K$

$L^* = S$ is a holonomorphic spin bundle

$d = \nabla + \Phi$

$$\phi = \Phi + (- \Phi^*)$$

$$\Phi \in \Gamma(\Sigma, K \otimes W)$$

$\nearrow$ $su(2)$ (1,0)-tem (0,1)-part of ϕ

$$= \Gamma(\Sigma, K \otimes \mathrm{End}_0(V))$$

$$\Phi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$$

$$\alpha \in \Gamma(\Sigma, K \underset{\substack{\shortparallel \\ K^{-1}}}{L^2}) = \Gamma(\Sigma, \mathbb{C})$$

$$\beta \in \Gamma(\Sigma, K \underset{\substack{\shortparallel \\ K}}{L^{-2}}) = \Gamma(\Sigma, K^2)$$

Computation yields:

$$\alpha = -\tfrac{1}{2} \cdot H \qquad \beta = -\tfrac{1}{2} \cdot Q \quad \text{Hopf diff.}$$

$f : \Sigma \to \mathbb{R}^3 \quad \rightsquigarrow \quad \mathrm{I} = g$

II - 2. nd fundam form

$$\underset{(2,0)}{\mathrm{II}} =: Q + \underset{(1,1)}{H \cdot g} + \underset{(0,2)}{\overline{Q}}$$

Hod. diff

$H := \frac{1}{2} \mathrm{tr}\, \mathrm{II}$

mean curv

Gauß - Codazzi $\qquad d = \nabla + \phi \quad$ is flat

$$0 = F^{(\nabla + \phi)} = \underbrace{F^{\nabla}}_{\text{diagonal}} + \underbrace{d^{\nabla}\phi}_{\text{off-diag.}} + \underbrace{\tfrac{1}{2}[\phi \wedge \phi]}_{\text{diagonal}}$$

$V = L \oplus L^{\perp}$

On $L \quad (\Leftrightarrow L^{\perp})$

$$F^{\nabla^{L}} + \tfrac{1}{2}[\phi \wedge \phi] = 0 \iff$$

$$\nabla = \begin{pmatrix} \nabla^{L} & 0 \\ 0 & \nabla^{L^{\perp}} \end{pmatrix}$$

$$0 = K_{g}\, dA + H^{\perp} dA - |Q|^{2}$$

Gauß - equation

$$V = \begin{pmatrix} V^L & 0 \\ 0 & V^{L*} \end{pmatrix} \qquad \Phi = \bar{\Phi} - \bar{\Phi}^{*}$$

$$\shortparallel$$

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\beta^{*} \\ -\alpha^{*} & 0 \end{pmatrix}$$

Codazzi eqn: $d^{\nabla} \Phi = 0$

$$\shortparallel$$

$$d^{\nabla} \bar{\Phi} - d^{\nabla} \bar{\Phi}^{*}$$

$$\Rightarrow \qquad g \, \partial H = \bar{\partial} Q$$

$$\Gamma(\bar{K} K K) \qquad \Gamma(\bar{K} K^{L})$$

Def: A CMC surface (minimal) is a conf. immersion $f: \Sigma \to \mathbb{R}^3$ s.th. $H = \frac{1}{2} \operatorname{tr} \underline{\mathrm{II}}$ is constant (0)

Rmk: Minimal surfaces are critical points of the area functional
CMC — " —
with fixed enclosed volume.

If $H = 0$ $\rightsquigarrow$ Q is holomorphic

pf: $\partial H = 0$

$\underset{\text{Codazzi}}{\rightsquigarrow}$ $\bar{\partial} Q = 0$

$$Q = q\,(dz)^2$$

$$\bar{\partial} Q = \frac{\partial q}{\partial \bar{z}}\, d\bar{z}\,(dz)^2$$

<u>Rmk</u>: $H = \text{const.} \iff Q$ is holo.

$\lambda \in S^1 \subseteq \mathbb{C}^*$

rotate: $\quad Q \rightsquigarrow \lambda Q$

$\qquad\qquad g \rightsquigarrow g$

new Gauss-Codazzi data

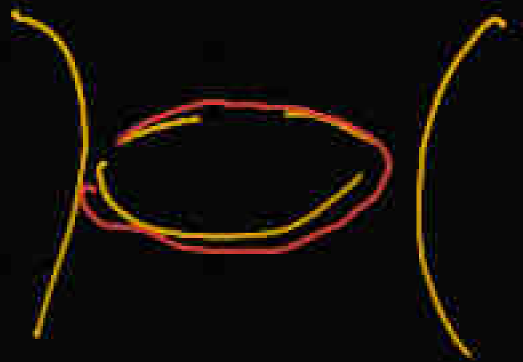
$\nabla + \Phi \rightsquigarrow \nabla + \tilde{\Phi}$

$$\tilde{\Phi} = \begin{pmatrix} 0 & \alpha \\ \lambda\beta & 0 \end{pmatrix} - \begin{pmatrix} 0 & \bar{\lambda}\beta^* \\ \alpha^* & 0 \end{pmatrix}$$

Observation: we obtain a new surface from the new solution $\nabla, \tilde{\phi}$ of the Gauß-Codazzi data

Example: $H=0$ associated family of minimal surfaces

Catenoid $\rightarrow$ Helicoid



Weierstraß-rep: für $H=0$, $\alpha=0$ suchen:

$$\Phi = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \qquad \Phi^* = \begin{pmatrix} 0 & \beta^t \\ 0 & 0 \end{pmatrix}$$

$$\nabla = \underline{\underline{L}} \oplus L^{-1}$$

$$d = \nabla + \phi$$

$$d^{(0,1)} = \nabla^{(0,1)} + \phi^{(0,1)}$$

$$= \begin{pmatrix} (\nabla^{L})^{0,1} & -\beta^t \\ \underline{\underline{0}} & (\nabla^{L^{-1}})^{0,1} \end{pmatrix}$$

$i : L = S^* \hookrightarrow V = \mathbb{C}^2$

$i = \begin{pmatrix} s \\ t \end{pmatrix}$ $\qquad s, t \in H^0(\Sigma, S)$

$S^2 = K$

Then $(1,0)$-part of the diff. of the immersion $f : \Sigma \to su(2)$

$$\omega = \begin{pmatrix} s \cdot t & -s^2 \\ t^2 & -st \end{pmatrix} \in H^0(\Sigma, K\, sl(2,\mathbb{C}))$$

$Q = \begin{pmatrix} \partial s \\ \partial t \end{pmatrix} \wedge \begin{pmatrix} s \\ t \end{pmatrix} = (\partial s)\, t - (\partial t)\, s$

Minimal surfaces in S^3

$$S^3 = SU(2) = \left\{ \begin{pmatrix} x & -\bar{y} \\ y & \bar{x} \end{pmatrix} \;\middle|\; \begin{matrix} |x|^2 + |y|^2 = 1 \\ x, y \in \mathbb{C} \end{matrix} \right\}$$

cpt Lie group with bi-invariant metric

$$\langle X, Y \rangle = \frac{1}{2}\, \underline{\mathrm{tr}\,(X\, \bar{Y}^t)}$$

LC connection $\quad \nabla = \frac{1}{2} \nabla^L + \frac{1}{2} \nabla^R$

$$= d + \frac{1}{2}\, g^{-1} dg$$

acting by adj. rep on $su(2)$.

minimal surface $\Leftrightarrow$ $f : \Sigma \to S^3$ conf. immersion

$+$ harmonic

MC equ's: $d^\nabla df = 0 \Leftrightarrow d(f^{-1}df) + \frac{1}{2}[f^{-1}df \wedge f^{-1}df] = 0$

harmonicity: equ's: $d^\nabla * df = 0 \Leftrightarrow 0 = d(* f^{-1}df) + \frac{1}{2}[\cancel{f^{-1}df \wedge * f^{-1}}df]$

$\Updownarrow$ f is conf.

$H = 0$

$* f^{-1}df$ is the diff. of a CMC surface in $\mathbb{R}^3$! $\underline{* f^{-1}df \in \Omega^1(\Sigma, su(2))}$