Rational curves in projective space

Rational curve $\mathbb{P}^1 \longrightarrow \mathbb{P}^n$

Such a map is given by

$$[f_0(s,t), \dots, f_n(s,t)]$$

$(n+1)$ polynomials of degree d with no common roots.

Examples

1) $\mathbb{P}^1 \longrightarrow \mathbb{P}^n$

$(s,t) \longmapsto (s,t,0,\dots,0)$

line $\quad x_2 = \dots = x_n = 0$

2) $\mathbb{P}^1 \longrightarrow \mathbb{P}^n$

$(s,t) \longmapsto (s^2, st, t^2, 0, \dots, 0)$

Conic $\quad x_3 = \dots = x_n = 0$

$$x_0 x_2 - x_1^2 = 0$$

3) $\mathbb{P}^1 \longrightarrow \mathbb{P}^n$

$(s,t) \longmapsto (s^3, s^2t, st^2, t^3, 0, \dots, 0)$

twisted cubic $\quad x_4 = \cdots = x_n = 0$

$$x_0 x_2 - x_1^2 = x_1 x_3 - x_2^2$$
$$= x_0 x_3 - x_1 x_2 = 0$$

dim space of rational curves of degree d in $\mathbb{P}^n$

$$\boxed{(n+1)(d+1) - 4}$$

There are two useful compactifications.

– <u>The Hilbert scheme</u>: Take the locus of smooth rational curves and take the closure in the Hilbert scheme

$$\mathcal{H}_{dt+1}(\mathbb{P}^n).$$

This gives a projective scheme containing smooth rational curves.

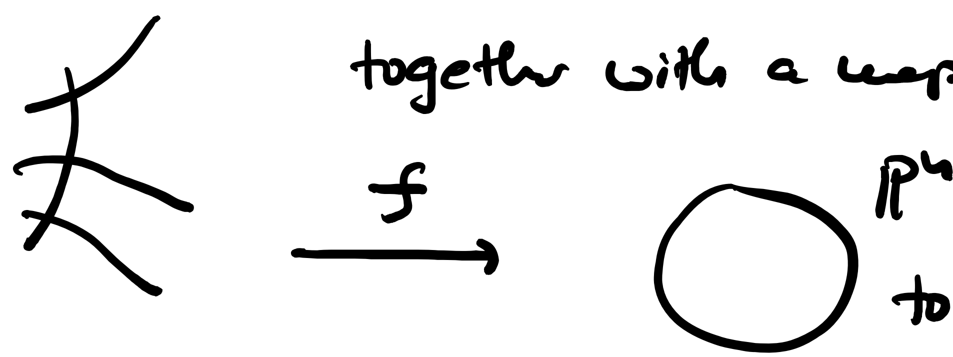
<u>Problem</u>: Often reducible, the boundary not well-understood.

– <u>Kontsevich moduli space</u>:

$\mathcal{M}_{0,0}(\mathbb{P}^n, d)$ parametrizes maps $\{(C, f) \mid$ C is a connected, nodal, reduced curve of arithmetic genus 0 and f is a map $f: C \to \mathbb{P}^n$ st. $f_*[C] = d\ell$ and any component of C contracted by f has at least 3 nodes$\}$

$\uparrow$ line class

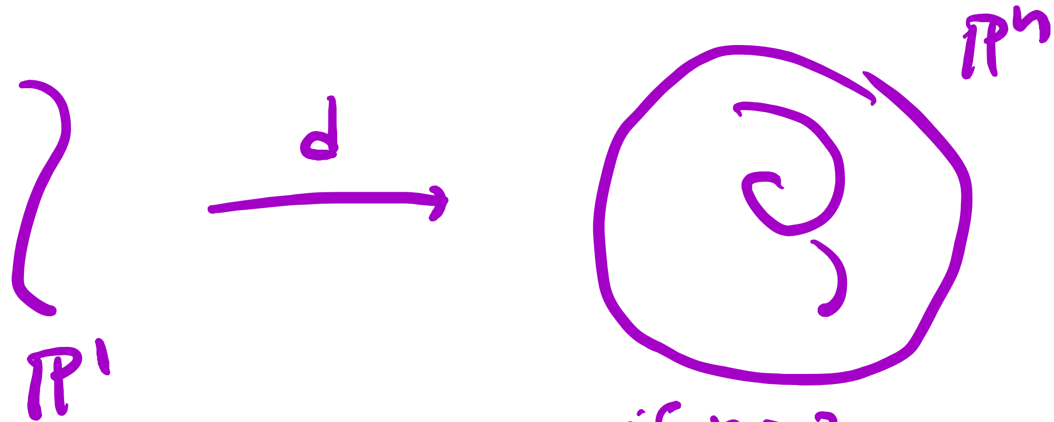
Points of this space are trees of rational curves



together with a map to $\mathbb{P}^n$

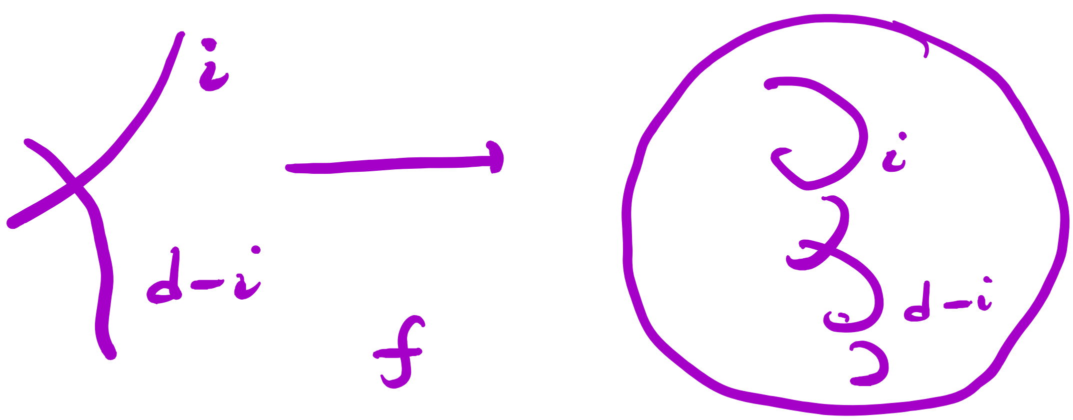
If f is constant on a component, then that component should have at least 3 nodes.

The general point of the space

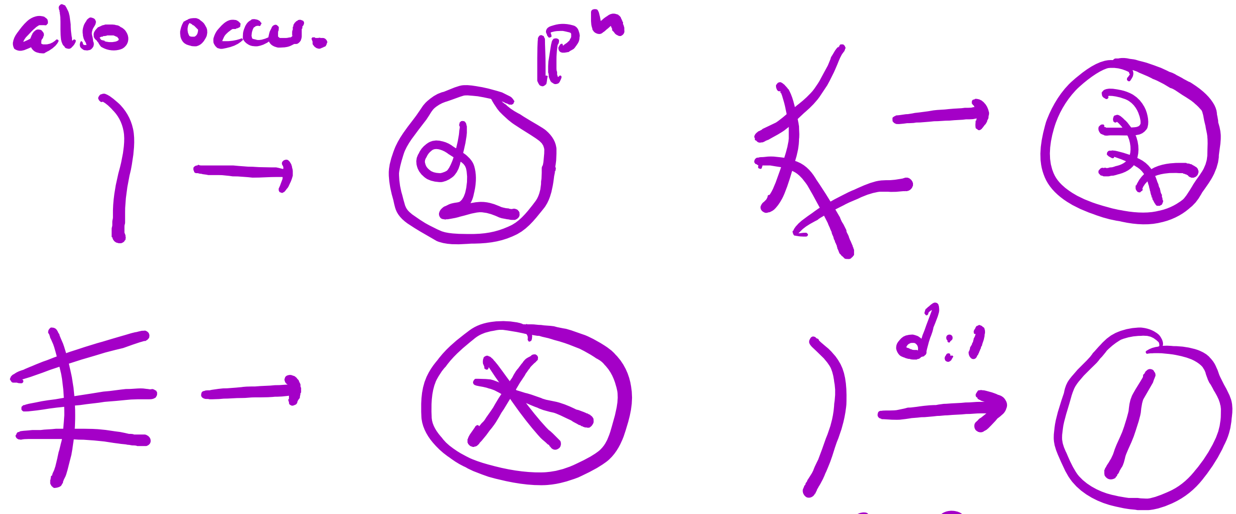


if $n > 2$ is an embedding of a smooth rational curve.

For each $1 \leq i \leq \frac{d}{2}$, there is a boundary divisor



Of course, more complicated behavior also occurs.



map d:1 onto a line.

Examples

1) $d=1$

lines in $\mathbb{P}^n$, $\quad \mathbb{G}(1,n)$

2) $d=2$

conics in $\mathbb{P}^n$.

The general point of either space is a smooth conic.

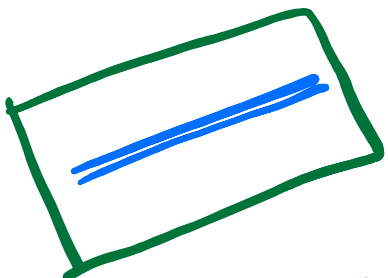
The Hilbert scheme

$$\mathrm{Hilb}_{2t+1}(\mathbb{P}^n) \simeq \begin{array}{c} \mathbb{P}\,\mathrm{Sym}^2 S^* \\ \downarrow \\ \mathbb{G}(2,n) \end{array}$$

you choose a plane and a curve of degree 2 in the plane.

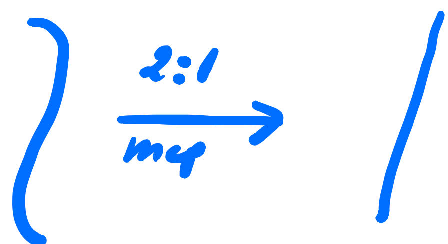
Kontsevich space differs from the Hilbert scheme along the locus of double lines.

<u>Hilbert scheme</u>



You need to specify the plane that contains it.

<u>Kontsevich</u>



onto a line

specify the 2 branch points.

3) $d=3$

The Hilbert scheme $\mathrm{Hilb}_{3t+1}(\mathbb{P}^n)$ is no longer irreducible

twisted cubic

<u>Thm.</u> $M_{0,0}(\mathbb{P}^n, d)$ is irreducible.

More generally, Kim and Pandharipande

prove that

$\overline{M}_{0,m}(G/P, \beta)$ is irreducible.

For homogeneous varieties, the Kontsevich space is a useful compactification of the space of smooth rational curves.

The tangent space

$H^0(N_{C/\mathbb{P}^n})$, $H^0(N_f)$

$f : C \longrightarrow \mathbb{P}^n$

$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^n}|_C \longrightarrow N_{C/\mathbb{P}^n} \to 0$

$0 \longrightarrow T_{\mathbb{P}^1} \longrightarrow f^* T_{\mathbb{P}^n} \longrightarrow N_f \to 0$

<u>Basic task</u>: Understand normal bundles.

Thm. (Birkhoff-Grothendieck) Every vector bundle on $\mathbb{P}^1$ is a direct sum of line bundles.

i.e. $\exists$ $a_1 \leq a_2 \leq \dots \leq a_r$ st.

$E \simeq \oplus \mathcal{O}_{\mathbb{P}^1}(a_i)$ and the sequence of a_i is unique.

Uniqueness is clear. If $\oplus \mathcal{O}(a_i) \simeq \oplus \mathcal{O}(b_i)$ say $a_j < b_j$ the largest integer that is diffent. Compute $E(-b_j)$ for both sides to get a contradiction.

Sketch of proof: Let a be the maximal integer st. $\exists$ nonzero map

$$\mathcal{O}_{\mathbb{P}^1}(a) \xrightarrow{\varphi} E$$

Note: φ is nowhere 0.

So: $0 \longrightarrow \mathcal{O}(a) \longrightarrow E \longrightarrow F \longrightarrow 0$

F is a vector bundle of rank $r-1$.

By induction on the rank

$$0 \longrightarrow \mathcal{O}(a) \longrightarrow E \longrightarrow \bigoplus^{r-1} \mathcal{O}(a_i) \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}(a) \longrightarrow E \longrightarrow \bigoplus_{i=1} \mathcal{O}(a_i) \longrightarrow 0$$

$$E \in \operatorname{Ext}^1\left(\bigoplus_{i=1}^{r-1} \mathcal{O}(a_i), \mathcal{O}(a)\right)$$

Need to show extension is trivial.

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}(-1) & \longrightarrow & E(-a-1) & \longrightarrow & \bigoplus_{i=1}^{r-1} \mathcal{O}(a_i-a-1) & \\ & & & & & \downarrow & \\ & & & & & 0 & \end{array}$$

$$H^0(E(-a-1)) = 0, \quad H^0(\mathcal{O}(-1)) = \underset{\substack{\| \\ 0}}{H^1(\mathcal{O}(-1))}$$

$$\Longrightarrow \quad H^0(\mathcal{O}(a_i-a-1)) = 0$$

$$\Longrightarrow \quad a_i - a - 1 < 0.$$

$$\boxed{a_i \leq a}.$$

$$\operatorname{Ext}^1(\mathcal{O}(a_i), \mathcal{O}(a)) = H^1(\mathcal{O}(a-a_i))$$
$$= 0.$$

$$E \simeq \bigoplus_{i=1}^{r-1} \mathcal{O}(a_i) \oplus \mathcal{O}(a). \qquad \square.$$

Given a vector bundle on $\mathbb{P}^1$ we can talk about the <u>splitting type</u>

$$E \simeq \bigoplus \mathcal{O}(a_i)$$

$$a_1 \leq a_2 \leq \cdots \leq a_r$$

The vector bundle is <u>balanced</u> if $|a_i - a_j| \le 1 \quad \forall\ i,j.$

<u>perfectly balanced</u> if $a_1 = a_2 = \cdots = a_r$

this requires $r \mid$ degree.

Given a smooth rational curve in a variety we have two important bundles

$$C \cong \mathbb{P}^1 \subset X$$

$T_X|_C$ and $N_{C/X}$

the tangent bundle of X restricted to C

normal bundle of C in X.

<u>Question</u>: What are the splitting types of these bundles?

It will be more convenient to think

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^n$$

$$0 \to T_{\mathbb{P}^1} \xrightarrow{df} f^* T_{\mathbb{P}^n} \longrightarrow N_f \to 0$$

$$\deg(T_{\mathbb{P}^1}) = 2 \qquad \deg(f^* T_{\mathbb{P}^n}) = \underset{\substack{\uparrow \\ \text{degree of curve}}}{d}(n+1)$$

$$\deg(N_f) = d(n+1) - 2$$

$$\mathrm{rk}(N_f) = n-1$$

$$N_f \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(d + b_i)$$

$$\boxed{\sum b_i = 2d - 2}$$

Versal deformation space

One would expect a bundle on $\mathbb{P}^1$ to be as balanced as possible.

Given a splitting type $E = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$ the codimension of this locus in the versal deformation space is given by

$$h^1(End(E)) = \sum_{\{i,j \mid a_i - a_j \leq -2\}} (a_j - a_i - 1)$$

<u>Question</u>: Do given splitting types occur in expected codimension?

Example: Monomial curves

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^n$$

$$\Big(\overset{s^{i_0}}{\overset{\|}{s^d}},\ \overset{s^{i_1}t^{d-i_1}}{\overset{\|}{s^{d-1}t}},\ s^{i_2}t^{d-i_2}, \dots, \overset{s^{i_{n-1}}t^{d-i_{n-1}}}{\overset{\|}{st^{d-1}}}, t^d\Big)$$

Then $\displaystyle N_f \simeq \bigoplus_{j=1}^{n-1} \mathcal{O}_{\mathbb{P}^1}(d + i_{j+1} - i_{j-1})$

$$\begin{tikzcd}
 & \mathcal{O} \arrow[r, phantom, "\simeq"] \arrow[d, "i"'] \arrow[dr, phantom, "\bigstar"] & \mathcal{O} \arrow[d, "\rho"] & & \\
0 \arrow[r] & \mathcal{O}(1)^2 \arrow[r, "M"] \arrow[d] & \mathcal{O}(e)^{n+1} \arrow[r] \arrow[d] & N_f \arrow[r] \arrow[d, phantom, "\wr\|"] & 0 \\
0 \arrow[r] & T_{\mathbb{P}^1} \arrow[r] \arrow[d] & f^*T_{\mathbb{P}^n} \arrow[r] \arrow[d] & N_f \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

This works when $\mathrm{char}(k) \nmid d$.

$\bigstar$ Euler relation

$$f = (f_0 : f_1 : \dots : f_n)$$

$$M \qquad \begin{pmatrix} \dfrac{\partial f_0}{\partial s} & \dfrac{\partial f_0}{\partial t} \\ \vdots & \end{pmatrix}$$

$$\begin{pmatrix} \vdots & \\ \frac{\partial f_n}{\partial s} & \frac{\partial f_n}{\partial t} \end{pmatrix}$$

We need to compute cokernel of

Euler relation

$$s\frac{\partial f_i}{\partial s} + t\frac{\partial f_i}{\partial t} = (\deg f_i) f$$

You can use this method to compute the normal bundle if $\mathrm{char}(k) \nmid d$.

<u>Caution</u>: If $\mathrm{char}\, k \mid d$, then this method does not work.

In fact, the normal bundle can depend on the characteristic.

For the rest of this lecture assume $\mathrm{char}(k) = 0$ or $\gg 0$

It is a little easier to dualize

$$0 \to N_f^* \longrightarrow \mathcal{O}(-e)^{n+1} \xrightarrow{M^t} \mathcal{O}(-1)^2 \to 0$$

In our case,

$$\begin{bmatrix} d s^{d-1} & (d-1) s^{d-2} t & i_2 s^{i_2-1} t^{d-i_2} & \cdots \\ 0 & s^{d-1} & (d-i_2) s^{i_2} t^{d-i_2-1} & \cdots \end{bmatrix}$$

There are relations among the triples of consecutive columns of this matrix

$$\begin{array}{ccc} i_j s^{i_j-1} t^{d-i_j} & i_{j+1} s^{i_{j+1}-1} t^{d-i_{j+1}} & i_{j+2} s^{i_{j+2}-1} t^{d-i_{j+2}} \\ (d-i_j) s^{i_j} t^{d-i_j-1} & (d-i_{j+1}) s^{i_{j+1}} t^{d-i_{j+1}-1} & (d-i_{j+2}) s^{i_{j+2}} t^{d-i_{j+2}-1} \\ \uparrow & \uparrow & \nearrow \\ t^{i_j-i_{j+2}} & s^{i_j-i_{j+1}} t^{i_{j+1}-i_{j+2}} & s^{i_j-i_{j+2}} \\ (i_{j+1}-i_{j+2}) & -(i_j-i_{j+2}) & (i_j-i_{j+1}) \end{array}$$

$$\bigoplus_{j=0}^{n-1} \mathcal{O}(-e-i_j+i_{j+2}) \longrightarrow \operatorname{Ker} M^t = N_f^*$$

$$\begin{bmatrix} -t^{i_2-i_0} & s^{\cdot} t^{\cdot} & -s^{i_2-i_0} & & \\ 0 & t^{i_3-i_1} & s^{\cdot} t^{\cdot} & s^{i_3-i_1} & \\ 0 & 0 & \ddots & & \\ & & & & \end{bmatrix}$$

every where injective map

there are two minors $t^{\cdot}$, $s^{\cdot}$

Injective map of vector bundles

$$\bigoplus_{j=0}^{n-1} \mathcal{O}(-e-i_j+i_{j+2}) \xrightarrow[\simeq]{} N_f^*$$

same rank and degree

So $N_f \simeq \bigoplus_{j=0}^{n-1} \mathcal{O}(e+i_j-i_{j+2})$

Rational normal curve in $\mathbb{P}^n$

$[s^n, s^{n-1}t, s^{n-2}t^2, \ldots, t^n]$

$$N_f \simeq \mathcal{O}(n+2)^{n-1}$$

<u>Rmk</u>. If a rational curve is degenerate $\quad C \subset \mathbb{P}^d \subset \mathbb{P}^n$

spans only a $\mathbb{P}^d$

$$0 \to N_{C/\mathbb{P}^d} \longrightarrow N_{C/\mathbb{P}^n} \longrightarrow N_{\mathbb{P}^d/\mathbb{P}^n}\big|_C \to 0$$

$$N_{C/\mathbb{P}^d} \cong \bigoplus_{i=1}^{d-1} \mathcal{O}(e+b_i) \qquad N_{\mathbb{P}^d/\mathbb{P}^n}\big|_C \cong \mathcal{O}(e)^{n-d}$$

Since $\mathrm{Ext}^1\left(\mathcal{O}(e)^{n-d}, \oplus\mathcal{O}(e+b_i)\right) = 0$

$$N_{C/\mathbb{P}^n} = N_{C/\mathbb{P}^d} \oplus \mathcal{O}(e)^{n-d}$$

So we can restrict ourselves to studying nondegenerate rational curves.

Sacchiero's construction

Thm. Let $b_i \geq 2$, $\sum_{i=1}^{n-1} b_i = 2e-2$

Then there exists an unramified map
$f: \mathbb{P}^1 \to \mathbb{P}^n$ of degree e
such that

$$N_f \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(e+b_i)$$

Think: All possible splitting types occur.
In particular, the generic splitting type is balanced.

Caution: This is false in char p!!!

Example: Let C be a general rational curve of degree $e = 2f$ in $\mathbb{P}^3$. Assume $\mathrm{char}(k) = 2$.

The Euler sequence in $\mathbb{P}^r$

$$0 \longrightarrow N_C^\vee(1) \longrightarrow \mathcal{O}_C^{\oplus r+1} \longrightarrow \mathcal{P}^1(\mathcal{O}_C(1)) \to 0$$

first bundle of principal parts.

$\pi : C \longrightarrow C^{(2)}$ Frobenius

$$\mathcal{P}^1(\mathcal{O}_C(1)) = \pi^* \pi_* \mathcal{O}_C(1)$$

$N_C^\vee(1)$ is the pullback of a vector bundle under Frobenius. So all summands are even.

$$N_C \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i) \qquad a_i \equiv d \mod 2$$

$N_C \simeq \mathcal{O}(2e-2) \oplus \mathcal{O}(2e)$ not balanced!

<u>Back to Sacchiero</u>:

Assume $b_1 \le b_2 \le \ldots \le b_{n-1}$

(all we need $b_{n-1} \ge \max(b_i | 1-1)$)

Let $\delta_1 = 1$

$$\delta_i = b_{i-1} - \delta_{i-1}$$

for $1 < i \le n-1$

$\sum^{n-1} \delta_i$

Set $c = 1 + \sum_{i=1} \delta_i$

Let $p(s,t)$ $q(s,t)$
be two polynomials of degree $e-c$
general enough [• no repeated roots
• no common roots
• not divisible by s or t]

Set

$$k_i = c - \sum_{j=1}^{i} \delta_j$$

So $k_0 = c$

$k_1 = c - \delta_1$

$k_2 = c - \delta_1 - \delta_2$

$\vdots$

Consider the map

$$f = (s^{k_0} p,\ s^{k_1} t^{c-k_1} p,\ s^{k_2} t^{c-k_2} p,$$

$$\ldots\ldots, s^{k_{n-1}} t^{c-k_{n-1}} p, t^c q)$$

$$f = \left(s^c p, s^{c-1} t p, \ldots, s^{k_{n-1}} t^{c-k_{n-1}} p, t^c q\right)$$

Compute Jacobian matrix

$$\begin{bmatrix} & k_i s^{k_i-1} t^{c-k_i} p + s^{k_i} t^{c-k_i} p_s & \cdots \\ \cdots & (c-k_i) s^{k_i} t^{c-k_i-1} p + s^{k_i} t^{c-k_i} p_t & \cdots \end{bmatrix}$$

The first $(n-2)$ columns satisfy easy relations

$$(k_i - k_{i+1}) t^{k_{i-1}-k_{i+1}}$$

$$-(k_i - k_{i+1}) s^{k_{i-1}-k_i} t^{k_i - k_{i+1}}$$

$$(k_{i-1} - k_i) s^{k_{i-1}-k_{i+1}}$$

$$\bigoplus^{n-2} \mathcal{O}(-e - k_{i-1} + k_{i+1}) \longrightarrow N_f^*$$

$i=1$

$$k_0 - k_2 = \delta_1 + \delta_2 = b_1$$

$$k_1 - k_3 = \delta_2 + \delta_3 = b_2$$

$$\vdots \qquad \text{etc.}$$

$$\bigoplus_{i=1}^{n-2} \mathcal{O}(-e-k_{i-1}-k_{i+1}) \longrightarrow N_f^*$$

$$\wr\wr$$

$$\bigoplus_{i=1}^{n-2} \mathcal{O}(-e-b_i) \longrightarrow N_f^* \longrightarrow \mathcal{O}(-e-b_{n-1})$$

injective map of v.6

$$\operatorname{Ext}^1\left(\mathcal{O}(-e-b_{n-1}), \bigoplus_{i=1}^{n-2} \mathcal{O}(-e-b_i)\right) = 0$$

As long as $b_{n-1} \geq b_i - 1$

$\forall\, i \leq n-2$

$$N_f \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(e+b_i)$$

Trouble in paradise:

$n \geq 6$ and assume

$$(n-2)(2e-2n-1) \geq (e+1)(n+1)$$

The expected codim of

$$\mathcal{O}(e+2)^{n-2} \oplus \mathcal{O}(3e-2n+2)$$

$$(n-2)(2e-2n-1)$$

By assumption, this is larger than dim Space of rational curves.

So we expect it to be empty

But.... it is not!!!

Thm (Miret) The locally closed locus in $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)$

parametrizing unramified morphisms when $f^* T_{\mathbb{P}^n}$ has the expected splitting type is irreducible of the expected dimension

$$(e+1)(n+1) - 1 - h^1(\mathrm{End}(f^* T_{\mathbb{P}^n}))$$

Pullback Euler sequence

$$0 \longrightarrow f^*\Omega_{\mathbb{P}^n} \longrightarrow \mathcal{O}(-e)^{n+1} \longrightarrow \mathcal{O} \to 0$$

The Kodaira-Spencer map

$$\mathrm{Hom}(\mathcal{O}(-e)^{n+1}, \mathcal{O})$$
$$\downarrow$$
$$\mathrm{Ext}^1(f^*\Omega_{\mathbb{P}^n}, f^*\Omega_{\mathbb{P}^n})$$

factors through natural morphisms

$$\mathrm{Hom}(\mathcal{O}(-e)^{n+1}, \mathcal{O}) \xrightarrow{1} \mathrm{Ext}^1(\mathcal{O}(-e)^{n+1}, f^*\Omega_{\mathbb{P}^n})$$

2

$\hookrightarrow \mathrm{Ext}^1(f^*\Omega_{\mathbb{P}^n}, f^*\Omega_{\mathbb{P}^n})$

① comes from $\mathrm{Hom}(\mathcal{O}(-1)^{n+1}, -)$

② comes from $\mathrm{Hom}(-, f^*\Omega_{\mathbb{P}^n})$

Both are surjective.

<u>Thm</u> (Eisenbud - Van de Ven
Ghione - Sacchiero)
The locus of f in $\mathrm{Mor}(\mathbb{P}^1, \mathbb{P}^3)$!!!
where the normal bundle has the specified splitting type is irreducible of the expected dimension.

<u>Conjecture</u> (Eisenbud - Van de Ven)
Same is true for $\mathbb{P}^n$.

We have already seen that these loci

do not always have the expected dimension.

Thm. (C- Riedl) Let $n \geq 3k-1$ and $e > 2k-2n-2$.

Then the locus where the splitting type has a subbundle

$$\mathcal{O}(e+2)^k \text{ has at least}$$

k irreducible components.

The construction is similar to Sacchiero's construction.

You consider degree 2 relations that look like

$$\begin{pmatrix} s^2 & -2st & t^2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & s^2 & -2st & t^2 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & s^2 & -2st & t^2 & 0 & 0 & \cdots & & & \\ 0 & 0 & 0 & 0 & 0 & s^2 & -2st & t^2 & 0 & \cdots & \end{pmatrix}$$

When $k=2$

$n \geq 5$ and $e \geq 2n-3$

$$\begin{pmatrix} s^2 & -2st & t^2 & 0 & 0 & \cdots & 0 \\ 0 & s^2 & -2st & t^2 & 0 & \cdots & 0 \end{pmatrix}$$

$$e(n-2)+5n-3$$

$$\begin{pmatrix} s^2 & -2st & t^2 & 0 & \cdots & & & & \\ 0 & 0 & 0 & s^2 & -2st & t^2 & 0 & \cdots \end{pmatrix}$$

$$e(n-3)+7n-6$$

In this case there are 2 components.

<u>Question</u> : Can we classify the irreducible components of the locus in $\mathrm{Mor}_e(\mathbb{P}^1, \mathbb{P}^n)^{ur}$ where the normal bundle has fixed splitting type?

So far we discussed rational curves in $\mathbb{P}^n$. We can ask the same questions for any smooth projective variety X.

- What is the dimension of the space of rational curves on X?
- Is the space irreducible?
- What is the generic splitting type of $f^* T_X$ or N_f?
- Which splitting types are possible?

<u>Example</u> (Sayanta Mandal)

$$G(k,n)$$

$$T_{G(k,n)} \simeq S^* \otimes Q$$

dual of the tautological

tautological subbundle $\qquad$ quotient bundle

$$\operatorname{rk} S^* = k \qquad \operatorname{rk} Q = n-k$$

$$\deg f^*S = e \qquad \deg f^*Q = e$$

If $k \nmid e$ and $n-k \nmid e$,

$f^*T_{G(k,n)}$ cannot be balanced.

$$C \subset G(2,4)$$

general deg 3 rational curve

$$S^*|_C \simeq \mathcal{O}(1) \oplus \mathcal{O}(2)$$

$$Q|_C \simeq \mathcal{O}(1) \oplus \mathcal{O}(2)$$

$$T_{G(2,4)}|_C \simeq \mathcal{O}(2) \oplus \mathcal{O}(3)^2 \oplus \mathcal{O}(4)$$

not balanced.