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Independent axioms for vector spaces

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In a recent paper [1], Victor Bryant shows how the number of axioms required to define a vector space can be reduced to seven (in addition to closure requirements). The main result of his article is that commutativity of addition can be deduced from the other axioms. In the present article we show how to reduce this number to six. For certain underlying fields one or more of these axioms can be deduced from the others. However, the six axioms are in general *independent*; we invite interested readers to show this by constructing their own counter-examples, which the editor of the *Gazette* will be pleased to receive.

The authors do not claim originality for all the proofs; those of theorems 1 and 2, for instance, employ standard techniques.

In an article such as this, dealing with the axiomatic basis of a subject, it is useful to emphasise that certain operations are distinct by using different symbols to denote them. Addition in a vector space is not usually the same thing as addition in the underlying field, nor is multiplication of a vector by a scalar usually the same thing as multiplication of two scalars; hence we use the slightly unfamiliar notation given below. Some such notation is not only useful but necessary when we come to construct counter-examples; this will be seen later in the article.

Let F be a field with zero 0 and identity 1; we shall denote addition and multiplication in F by $+$ and $\times$. Let V be a set on which are defined (i) a binary operation of addition, denoted by $\oplus$, under which V is closed, and (ii) a 'scalar multiplication' by elements of F, so that with each λ in F and each a in V there is associated an element λa in V. We shall consider the following axioms:

1. $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ for all a, b, c in V,
2. $\lambda(a \oplus b) = \lambda a \oplus \lambda b$ for all λ in F, all a, b in V,
3. $(\lambda + \mu) a = \lambda a \oplus \mu a$ for all λ, μ in F, all a in V,
4. $(\lambda \times \mu) a = \lambda(\mu a)$ for all λ, μ in F, all a in V,
5. $0a = 0b$ for all a, b in V,
6. $1a = a$ for all a in V.

We first prove two theorems that together show that if V satisfies all six axioms then it is a vector space over F.

THEOREM 1. *If V satisfies axioms,* **1, 3, 5, 6,** *then it is a group under* $\oplus$.

PROOF. Let z denote the constant element given by axiom **5**. Then, for all a in V,

$$a = 1a = (1 + 0)a = 1a \oplus 0a = a \oplus z;$$

thus z is a right neutral of addition. Next, for all a in V,

$$z = 0a = (1 + (-1))a = 1a \oplus (-1)a = a \oplus (-1)a;$$

thus a has $(-1)a$ as an additive right inverse relative to z. We can now either appeal to a general theorem to deduce that V is an additive group (see, e.g., [1]), or show directly that z is also a left neutral and that $(-1)a$ is also a left inverse of a.

THEOREM 2. *If V satisfies axioms* **1, 2, 3, 5, 6** *then it is a commutative additive group (i.e., a commutative group under* $\oplus$*).*

PROOF. For all a, b in V,

$$\begin{aligned} a \oplus a \oplus b \oplus b &= (1 + 1)a \oplus (1 + 1)b = (1 + 1)(a \oplus b) \\ &= 1(a \oplus b) \oplus 1(a \oplus b) = a \oplus b \oplus a \oplus b; \end{aligned}$$

since V is a group we deduce by cancellation that $a \oplus b = b \oplus a$.

As an immediate corollary of theorems 1 and 2, we have

THEOREM 3. *If V satisfies all six axioms, then it is a vector space over F.*

To show that an axiom is independent of the remaining axioms, we must find a counter-example not satisfying the axiom concerned but satisfying all the others. Such a counter-example serves to show that the axiom cannot be deduced from the others. We shall consider each axiom from this point of view, taking them in order of difficulty. We start by constructing a counter-example for axiom **6**, to encourage readers to construct their own for the various other axioms.

Axiom **6**. Given a field F, we construct an algebraic system V^* as follows. Let $V^* = F$; define $a \oplus b = a$ and $\lambda a = 0$ for all a, b in V^* and all λ in F. We can easily check that axioms **1–5** are satisfied in V^*; for instance

$$\begin{aligned} &(a \oplus b) \oplus c = a \oplus c = a, \quad (\lambda + \mu)a = 0, \\ &a \oplus (b \oplus c) = a \oplus b = a, \quad \lambda a \oplus \mu a = 0 \oplus 0 = 0. \end{aligned}$$

However, axiom **6** is not satisfied since $1a = 0 \neq a$ (unless $a = 0$).

As we mentioned earlier, if we were not using the special notation, the above definitions would become $a + b = a$, $\lambda \times a = 0$, which is confusing to say the least, and it would be impossible to see what was happening in the checking of the axioms.

We use the notation "V^*" here to emphasise that we are constructing a *particular* algebraic structure satisfying certain axioms, whereas we use "V" to denote the general structure satisfying certain axioms.

In this counter-example, addition in V^* is not commutative since $a \oplus b = a$ but $b \oplus a = b$, and V^* is not a group since there is no neutral element z in V^* such that $z \oplus a = a$ for all a in V^*. By changing the definition of $a \oplus b$, it is easy to construct a counter-example (again satisfying axioms **1–5** but not **6**) in which V^* *is* a commutative additive group. (See Hans Liebeck's article [3], but try it for yourself first; the article contains an interesting theorem about axiom **6**.)

It is natural to ask whether we can find (a) a counter-example in which V^* is a non-commutative additive group, and (b) a counter-example in which V^* is *not* an additive group but in which addition *is* commutative. Both types of counter-example can be constructed. Remember that in the counter-examples V^* will not usually be the same set as F.

We shall now consider in detail for each of the other axioms what sort of counter-example one can expect to construct not satisfying that axiom.

Axiom **5**. Counter-examples exist (satisfying axioms **1–4** and **6**, but not **5**) in which addition in V^* is either commutative or non-commutative. It is no use trying to construct a counter-example in which V^* is an additive group. For suppose that V is an additive group with neutral element z, satisfying axioms **1–4** and **6**; the reader may like to supply a proof that $0a = z$ for all a in V, so that axiom **5** is satisfied.

We can however produce a counter-example in which V^* has an additive neutral without being an additive group. Hence we cannot replace axiom **5** by the axiom

5*. V contains an additive neutral;

this axiom, together with axioms **1–4** and **6**, is not strong enough to ensure that V is a vector space.

Axiom **3**. By taking $V^* = F$ and defining addition in V^* and scalar multiplication suitably, counter-examples can be constructed in which addition in V^* is either commutative or non-commutative, but in which V^* is not an additive group.

Counter-examples also exist in which V^* is any given additive group.

Axiom **1**. Counter-examples exist in which addition in V^* is either commutative or non-commutative. The authors' own counter-examples are not trivial but require only a basic knowledge of vector spaces for their construction.

Axiom **2**. If all axioms except **2** are satisfied, V must be an additive group, by theorem 1.

(i) The following theorem is easily proved.

THEOREM 4. *If F is the field of only two elements,* 0 *and* 1, *axiom* **2** *is a consequence of the remaining axioms (in fact, a consequence of axioms* **3**, **5** *and* **6** *only).*

(ii) If F contains more than two elements, axiom **2** is not a consequence of the remaining axioms. In the counter-example constructed by the authors, V^* is not a *commutative* additive group unless F has characteristic 2 (i.e., unless $1 + 1 = 0$ in F). If F does have characteristic 2, V^* must be a commutative additive group in any counter-example, because we can easily prove

THEOREM 5. *If V satisfies axioms* **1**, **3**, **5** *and* **6**, *and if F has characteristic* 2, *then $a \oplus b = b \oplus a$ for all a, b in V.*

Can we produce a counter-example in which V^* *is* a commutative additive group, when the characteristic of F is not 2? The answer depends on F, as we show in (iii) and (iv) below.

(iii) In lemmas 1 and 2, and in theorem 6, we assume that *F is the rational field, and that V is any structure satisfying all the axioms except* **2**, *with the further assumption that addition in V is commutative.*

LEMMA 1. *If p is a positive integer, then $p(a \oplus b) = pa \oplus pb$ for all a, b in V.*

PROOF.

$$\begin{aligned} p(a \oplus b) &= (1 + 1 + \ldots + 1)(a \oplus b) \\ &= (a \oplus b) \oplus (a \oplus b) \oplus \ldots \oplus (a \oplus b) \\ &= (a \oplus a \oplus \ldots \oplus a) \oplus (b \oplus b \oplus \ldots \oplus b) \\ &= (1 + 1 + \ldots + 1)a \oplus (1 + 1 + \ldots + 1)b \\ &= pa \oplus pb. \end{aligned}$$

LEMMA 2. *If p is a negative integer, of if $p = 0$, then $p(a \oplus b) = pa \oplus pb$.*

PROOF. If $p = 0$ the proof is trivial. If p is negative, write $p = -p'$. Then $p(a \oplus b) \oplus p'(a \oplus b) = (p + p')(a \oplus b) = 0(a \oplus b) = z = z \oplus z$ by theorem 1, $= (p+p')\,a \oplus (p+p')\,b = pa \oplus p'\,a \oplus pb \oplus p'\,b = pa \oplus pb \oplus p'\,a \oplus p'b$ $= pa \oplus pb \oplus p'(a \oplus b)$ by lemma 1. We know that V is a group; hence by the cancellation law $p(a \oplus b) = pa \oplus pb$.

THEOREM 6. *$\lambda(a \oplus b) = \lambda a \oplus \lambda b$ for all λ in F, all a, b in V; i.e., when F is the field of rationals and addition in V is commutative then axiom* **2** *is a consequence of the remaining axioms.*

PROOF. Write $\lambda = p/q$, where p and q are integers, q positive. Then

$$\lambda(a \oplus b) = \left(\frac{1}{q} \times p\right)(a \oplus b) = \frac{1}{q}[p(a \oplus b)] = \frac{1}{q}[pa \oplus pb] \text{ by lemma 1 or 2,}$$

$$= \frac{1}{q}\left[\left(q \times \frac{p}{q}\right) a \oplus \left(q \times \frac{p}{q}\right) b\right] = \frac{1}{q}\left[q\left(\frac{p}{q} a\right) \oplus q\left(\frac{p}{q} b\right)\right] = \frac{1}{q}\left[q\left(\frac{p}{q} a \oplus \frac{p}{q} b\right)\right]$$

by lemma 1, $= \left(\frac{1}{q} \times q\right)\left(\frac{p}{q} a \oplus \frac{p}{q} b\right) = 1(\lambda a \oplus \lambda b) = \lambda a \oplus \lambda b.$

Next, if F is a field of prime order, then every element of F can be written in the form $1 + 1 + \ldots + 1$, and we show as in the proof of lemma 1 that axiom **2** is a consequence of the remaining axioms when addition in V is commutative.

(iv) Suppose F is not the rational field or a field of prime order. Using rather deep properties of field extensions we can show the existence of counter-examples in which V^* is a commutative additive group.

Axiom **4**. Suppose V satisfies all axioms except **4**. By theorem 2, V must be a commutative additive group.

(i) In lemmas 3, 4 and 5, and in theorem 7, we assume that F is the rational field.

LEMMA 3. *If p is a positive integer, then $p(\lambda a) = (p \times \lambda)a$ for all λ in F, all a in V.*

PROOF.

$$p(\lambda a) = (1 + 1 + \ldots + 1)(\lambda a) = \lambda a \oplus \lambda a \oplus \ldots \oplus \lambda a$$
$$= (\lambda + \lambda + \ldots + \lambda) a = (p \times \lambda) a.$$

LEMMA 4. *If p is a negative integer, or if $p = 0$, then $p(\lambda a) = (p \times \lambda)a$.*

PROOF. If $p = 0$ the result is trivial. If p is negative, write $p = -p'$. Then $p(\lambda a) \oplus p'(\lambda a) = (p + p')(\lambda a) = 0(\lambda a) = 0a$ by axiom **5**, $= (p \times \lambda + p' \times \lambda) a$ $= (p \times \lambda) a \oplus (p' \times \lambda) a = (p \times \lambda) a \oplus p'(\lambda a)$ by lemma 3. We know that V is an additive group by theorem 1: hence by the cancellation law $p(\lambda a) = (p \times \lambda) a$.

LEMMA 5. *If q is a positive integer, then $\frac{1}{q}(qa) = a$ for all a in V.*

PROOF.

$$\frac{1}{q}(qa) = \frac{1}{q}[(1 + 1 + \ldots + 1) a] = \frac{1}{q}(a \oplus a \oplus \ldots \oplus a)$$
$$= \frac{1}{q} a \oplus \frac{1}{q} a \oplus \ldots \oplus \frac{1}{q} a = \left(\frac{1}{q} + \frac{1}{q} + \ldots + \frac{1}{q}\right) a = 1a = a.$$

THEOREM 7. *$\lambda(\mu a) = (\lambda \times \mu)a$ for all λ, μ in F and all a in V; i.e., when F is the field of rationals then axiom* **4** *is a consequence of the remaining axioms.*

PROOF. Write $\lambda = p/q$, where p and q are integers, q positive. Then

$$\begin{aligned}\lambda(\mu a) &= \left(p \times \frac{1}{q}\right)(\mu a) = p\left(\frac{1}{q}(\mu a)\right) \quad \text{by lemma 3 or 4,}\\ &= p\left(\frac{1}{q}\left(q\left(\frac{\mu}{q}a\right)\right)\right) \quad \text{by lemma 3,}\\ &= p\left(\frac{\mu}{q}a\right) \quad \text{by lemma 5,}\\ &= \left(p \times \frac{\mu}{q}\right)a \quad \text{by lemma 3 or 4,}\\ &= (\lambda \times \mu)a.\end{aligned}$$

Next, if F is a field of prime order, then every element of F can be written in the form $1 + 1 + \ldots + 1$, and we show as in the proof of lemma 3 that axiom **4** is a consequence of the other axioms.

(ii) Next, suppose that F is the field $Q(\sqrt{2})$ of all real numbers of the form $\lambda + \lambda'\sqrt{2}$, where λ and λ' are rational. Let $V^* = Q$ (the rational numbers) and define $a \oplus b = a + b$, $(\lambda + \lambda'\sqrt{2})a = \lambda \times a$. Then $\sqrt{2}(\sqrt{2}(1)) = \sqrt{2}(0)$ (writing $\sqrt{2}$ in the form $0 + 1\sqrt{2}) = 0$, but $(\sqrt{2} \times \sqrt{2})(1) = 2(1) = 2$. Hence V^* does not satisfy axiom **4**. However, V^* satisfies the other axioms, as we may easily check.

This is the simplest case of counter-examples that always exist whenever F is not the rational field or a field of prime order. To show the existence in general of such counter-examples, the authors have had to use results about the decomposability of infinite groups ([2], pp. 122, 163).

Conclusions. (i) If F is the field of two elements, then axioms **2** and **4** are consequences of the other four independent axioms.

(ii) If F is any other field of prime order, or if F is the rational field, then axiom **4** is a consequence of the other five independent axioms.

(iii) If F is any other field, then the six axioms are independent.

(iv) All four axioms **1, 3, 5** and **6** are necessary for the proof that V is an additive group.

(v) All axioms except **4** are necessary for the proof that V is a commutative additive group, unless F has characteristic 2, when axiom **2** is not necessary.

(vi) If F is a field of prime order, or if F is the rational field, then, assuming axioms **1, 3, 4, 5** and **6**, and also assuming that addition in V is commutative, we can deduce axiom **2**.

(vii) We cannot replace axiom **5** by the axiom "V contains an additive neutral".

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Classroom notes

280. Disintegration

A recent A-level script produced an attempt that, even after some 35 years of examining, I found entirely new:

$$\text{“}\int \frac{dx}{x^2} = \int \frac{dx}{x^2 + 0^2} = \frac{1}{0}\tan^{-1}\left(\frac{x}{0}\right).\text{”}$$

Salvage is interesting:

$$\int_k^x \frac{dt}{t^2 + \varepsilon^2} = \frac{1}{\varepsilon}\left\{\tan^{-1}\left(\frac{x}{\varepsilon}\right) - \tan^{-1}\left(\frac{k}{\varepsilon}\right)\right\}$$

$$= \frac{1}{\varepsilon}\tan^{-1}\left\{\frac{(x/\varepsilon) - (k/\varepsilon)}{1 + (xk/\varepsilon^2)}\right\}$$

$$= u_\varepsilon, \text{ say.}$$

(Assume $k > 0$ for convenience.) Then

$$\varepsilon u_\varepsilon = \tan^{-1}\left\{\frac{\varepsilon(x-k)}{\varepsilon^2 + xk}\right\},$$

so that

$$\tan(\varepsilon u_\varepsilon) = \frac{\varepsilon(x-k)}{\varepsilon^2 + xk},$$

or

$$\frac{\sin(\varepsilon u_\varepsilon)}{\varepsilon} = \frac{(x-k)\cos(\varepsilon u_\varepsilon)}{\varepsilon^2 + xk}.$$

In the limit,

$$u = \frac{x-k}{xk} = \left[-\frac{1}{t}\right]_k^x = -\frac{1}{x} + \text{constant.}$$

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