IMBEDDING OF INFINITE DIMENSIONAL DISTRIBUTIONS INTO

SIMPLIFIED COLOMBEAU TYPE ALGEBRAS

by

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Abstract. Using a d-dimensional Gaussian probability space, a simplified Colombeau-type algebra is constructed containing the Meyer-Watanabe distributions. A parallel construction starting by a certain Gelfand triplet also includes the Hida distributions. As an application a new interpretation of the Feynman integrand as a generalized function in our sense is proposed.

Keywords. Asymptotic Colombeau extension, Meyer-Watanabe distributions, Hida distributions, Donsker's delta function, sharp uniform topology, Feynman integrand.

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1 Introduction

It is well-known that the Colombeau algebra developed in 1984 contained the distribution spaces $\mathcal{D}'$ and $\mathcal{S}'$ ([1], [2], [3], [4]) and warded off the consequences of the Schwartz's Impossibility Result and Stability Paradox in an optimal manner, i.e. the concessions given were kept at a minimum. In return there was a well-defined rule of multiplication for distributions. In a simplified version of the Colombeau algebra, the functional index set which is an essential feature in Colombeau's theory is avoided at the expense of non-existence of a privileged inclusion of $\mathcal{D}'$ distributions. However infinite number representatives are related within equivalence relations.

Since then various aspects, extensions and ramifications of the Colombeau's theory have been investigated including its random version , use in the solution of non-linear p.d.e. and relation to the non-standard analysis and ultra-distributions, ([5], [6], [7], [8],[9], [10], [11], [12], [13], [14]).
However there have not been noticable attempts to construct associative, commutative algebras containing infinite dimensional distributions with a possible exception of [15] where Hida distributions are imbedded into an algebra via chaos expansions.

In Section 2 , inspired by the Colombeau a-extension proposed by Delcroix & Scarpalezos ([16]) we start by a d-dimensional Gaussian space and construct an asymptotic, polynomial scale Colombeau extension $\mathcal{H}_s$. We show that the Meyer-Watanabe distributions $\mathbb{D}^{-\infty}$ are included in the algebra $\mathcal{H}_s$.
In Section 3 we consider a particular Gelfand triplet $E \subset H \subset E^*$ and the related Hida distribution. Using the Gaussian measure μ on E^* provided by the Minlos theorem we construct a parallel of the algebra $\mathcal{H}_s$ which also contains the Hida distributions. Thus in $\mathcal{H}_s$ the product of Hida distributions is well-defined in comparison to the indirect and somewhat artificial Wick product via the inverse $S-$transform.
After extending the results to the complex case in subsection 3.2, we give as an application, a new interpretation of the Feynman integrand as a distribution in our sense in 3.3.

The present article is based on and the extension of the results taking place in unpublished talks given by the author in international conferences [17], [18].

2 Inclusion of Meyer-Watanabe Distributions

2.1 Basic Framework and Notation

$\Omega = C_0([0,1])$: The set of all $\mathbb{R}^d-$valued continuous functions on $[0,1]$, null at zero with the sup norm.
μ : the Wiener measure on $(\Omega, \mathcal{B}(\Omega))$.
For $t \in [0,1], \omega \in \Omega, , W_t(\omega) \doteq \omega(t)$ is an $\mathbb{R}^d-$valued Wiener functional and $W(t,\omega)$ is a d-dimensional Wiener process.
$\mathcal{F}_t : t \in [0,1]$: the natural filtration generated by the process. $\mathcal{F} \equiv \mathcal{F}_\infty$ is the $\mu-$completion of the sigma algebra $\mathcal{B}(\Omega)$, so that $(\Omega, \mathcal{F}, \mu)$ is our Wiener space.
H : the Cameron-Martin (C-M) space , is the Hilbert space formed by all $h \in \Omega$ such that each component of $(h(t) = (h_1(t), h_2(t), \cdots, h_d(t))$ is absolutely continuous and has square integrable derivatives.
The inner product in H : for $g,\, h \in H$

$$(g,h)_H = \int_0^1 \sum_1^d \dot{g}_i(t)\, \dot{h}_i(t)\, dt \text{ and the Hilbertian norm } |h|_H^2 = \int_0^1 |\dot{h}(t)|^2 dt \tag{2.1.1}$$

An $\mathcal{F}$-measurable fuction $F : \Omega \to \mathbb{R}$ is a Wiener functional. Polynomial (resp. smooth) functionals are denoted by $\mathcal{P}$ (resp.S_M) We have $\mathcal{P} \subset S_M \subset L^p \equiv L^p(\Omega, \mathcal{F}, \mu)$, also $\mathcal{P}$ is dense in L^p, $(1 \le p < \infty)$.
Ocassionally we use E-valued functionals $F : \Omega \to E$, where E is a separable Hilbert space. In this case the symbols are $\mathcal{P}(E) \subset S_M(E) \subset L^p(E)$.
D_h denotes the perturbation of Wiener functionals in the direction of members of the C-M space, i.e. the weak derivative defined by the well-known formula $D_h F(\omega) = \lim_{\lambda \to 0} \dfrac{F(\omega + \lambda h) - F(\omega)}{h}$; $h \in H,\, F \in S_M$.
The gradient operator D is the closable operator from L^p into $L^p(H)$ determined by

$$(DF, h)_H = D_h F;\ DF \in S_M(H) \tag{2.1.2}$$

For E-valued functionals, the counterpart of (2.1.2) is

$$(DF, h \otimes e)_{H \otimes E} = \lim_{\lambda \to 0} \frac{1}{\lambda}(F(\omega + \lambda h) - F(\omega),\, e)_E\, ; e \in E,\, DF \in S_M(H \otimes E) \tag{2.1.3}$$

The chain rule for the gradients is

$$D(\,\phi(F_1, F_2, \cdots F_n) = \sum_{j=1}^n \partial_j \phi(F_1, F_2, \cdots F_n)\, DF_i \tag{2.1.4}$$

where $F_i \in S_M$, $(i = 1, \cdots, n)$ and ϕ is a tempered $\mathcal{C}^\infty$-function on $\mathbb{R}^n$, (in fact it suffices that ϕ is smooth , c.f. ([19], II, (2.26))
The Ornstein-Uhlenbeck operator is $\mathcal{L} = -\delta\, D$ where δ is the divergence operator, i.e. the adjoint of D.
The norms defined on polynomial functionals: $\|F\|_{s,p} = \|(I - \mathcal{L})^{s/2} F\|_p$, $s \in \mathbb{R}, 1 < p < \infty$ have the properties of monotonicity and consistency. It is known that for $s = k \in \mathbb{N}$

$$\|F\|_{k,p} = \left(\|F\|_{L^p}^p + \sum_{j=1}^k \|D^j F\|_{L^p(\mu; H^{\otimes j})}^p \right)^{\frac{1}{p}} \tag{2.1.5}$$

where $D^j F = D\,(D^{j-1}F), \in S_M(H^{\otimes j})$, (the tensor products are symmetrized).
We denote by $\mathbb{D}_s^p$, the Banach space which is the completion of S_M with respect to the norm $\|\,.\,\|_{s,p}$ and

$$\mathbb{D}^\infty = \bigcap_{s>0} \bigcap_{1<p<\infty} \mathbb{D}_s^p, \quad \mathbb{D}^{-\infty} = \bigcup_{s>0} \bigcup_{1<p<\infty} \mathbb{D}_{-s}^p \tag{2.1.6}$$

are Meyer-Watanabe testing functional space and its dual space (Meyer-Watanabe distributions) respectively.
$\mathbb{D}^\infty$ is a complete countably normed space (Frechet space) and also a topological algebra. By the monotonicity of $\|\,.\,\|_{s,p}$ norms we also have

$$\mathbb{D}^\infty = \bigcap_{k \in \mathbb{N}} \bigcap_{1<p<\infty} \mathbb{D}_k^p \tag{2.1.7}$$

Natural generalizations to E-valued functionals would be

$$\mathbb{D}^{\infty}(E) = \bigcap_{s>0} \bigcap_{1<p<\infty} \mathbb{D}^{p}_{s}(E), \ \mathbb{D}^{-\infty}(E) = \bigcup_{s>0} \bigcup_{1<p<\infty} \mathbb{D}^{p}_{-s}(E)$$

In many respects the pair $(\mathbb{D}^{\infty}, \mathbb{D}^{-\infty})$ behaves like $(\mathcal{D}, \mathcal{D}')$ of the Schwartz theory.

2.2 Construction of the Algebra $\mathcal{H}_s$

Let χ be the set of all functions from $(0,1]$ into $\mathbb{D}^{\infty}$. A member of χ will then be denoted by $(f_\epsilon)_\epsilon \in \chi \equiv (\mathbb{D}^{\infty})^{(0,1]}$.

Definition 2.2.1. $(f_\epsilon)_\epsilon \in \chi$ is called moderate if given $(k,p),\ k \in \mathbb{N}, 1 < p < \infty,\ \exists N \in \mathbb{N}$ such that $\|f_\epsilon\|_{k,p} = O(\epsilon^{-N})$ as $\epsilon \to 0$.
The set of all moderate elements in χ is denoted by $\mathcal{M}_s = (\Omega, \mathcal{F}, \mu; H)$.

Definition 2.2.2. $(f_\epsilon)_\epsilon \in \chi$ is called negligible (or null), if given $(k,p),\ k \in \mathbb{N},, 1 < p < \infty,\ \exists N \in \mathbb{N}$ such that $\|f_\epsilon\|_{k,p} = O(\epsilon^{m-N}),\ \forall m \geq N$ as $\epsilon \to 0$.
The set of all negligible members of χ is denoted by $\mathcal{J}_s(\Omega, \mathcal{F}, \mu; H)$.

Definition 2.2.3. $\mathcal{H}_s = \mathcal{M}_s(\Omega, \mathcal{F}, \mu; H)/\mathcal{J}_s(\Omega, \mathcal{F}, \mu; H)$ will be called the Colombeau extension of $\mathbb{D}^{\infty}$, can also be called the space of Wiener-Colombeau distributions).
(Note. The subindex 's' is the reminiscent of the word 'simplified' . For $\mathbb{D}^{\infty}$ being replaced by $\mathcal{C}^{\infty}(\Omega);\ \Omega \subset \mathbb{R}^n$ and $\|\,.\,\|_{k,p}$ norms by single index semi-norms in terms of supremums of partial derivatives on increasing compacts we retrieve the simplified Colombeau generalized functions, [16]).

Definition 2.2.4. Two members $(f_\epsilon)_\epsilon,\ (g_\epsilon)_\epsilon$ of χ are said to be ϵ-equivalent if $\lim_{\epsilon\to 0}(f_\epsilon - g_\epsilon) = 0$.

Definition 2.2.5. Two members $(f_\epsilon)_\epsilon,\ (g_\epsilon)_\epsilon$ of χ are said to be law equivalent if they have the same probability distribution.

Conclusions.
1. $\mathcal{M}_s$ is an algebra : Let $(f_\epsilon)_\epsilon,\ (g_\epsilon)_\epsilon \in \mathcal{M}_s$, define :
$(f_\epsilon)_\epsilon \,.\, (g_\epsilon)_\epsilon \doteq (f_\epsilon \,.\, g_\epsilon)_\epsilon \in \chi$ since $\mathbb{D}^{\infty} \equiv \mathbb{D}^{\infty}(\mathbb{R})$ is an algebra. Given $(k,p),\ k \in \mathbb{N},\ 1 < p < \infty,\ \exists N_1,\ N_2 \in \mathbb{N},\ \|f_\epsilon\|_{k,p} = O(\epsilon^{-N_1})$ and $\|g_\epsilon\|_{k,p} = O(\epsilon^{-N_2})$. Then $\|f_\epsilon \,.g_\epsilon\|_{[k,p} = O(\epsilon^{-N_1-N_2})$, hence $(f_\epsilon.g_\epsilon)_\epsilon \in \mathcal{M}_s$.

2. $\mathcal{J}_s$ is an ideal in $\mathcal{M}_s$: Let $(f_\epsilon)_\epsilon \in \mathcal{J}_s$ and $(g_\epsilon)_\epsilon \in \mathcal{M}_s$. Given (k,p) as in 1. $\exists N_1, N_2 \in \mathbb{N},\ \|f_\epsilon\|_{k,p} = O(\epsilon^{m-N_1})$ $\forall m \geq N_1,\ \|g_\epsilon\|_{k,p} = O(\epsilon^{-N_2})$. Then $\|f_\epsilon \,.\, g_\epsilon\|_{k,p} = O(\epsilon^{m-N})\ \forall m \geq N,$ where $N = N_1 + N_2$.

3. $\mathcal{H}_s$ is a factor algebra. If $F \in \mathcal{H}_s,$ it is of the form $(f_\epsilon)_\epsilon + \mathcal{J}_s,\ (f_\epsilon)_\epsilon \in \mathcal{M}_s$.

4. By the monotonicity of the norms $\|\,.\,\|_{k,p}$, if the conditions in Definitions 2.2.1 and 2.2.2 hold for a certain pair (k,p), then they also hold for all pairs in the cone $\{(k',p') :\ k' \leq k,\ p' \leq p\}$.

Inclusions.

A) $\mathbb{D}^{\infty} \subset \mathcal{M}_s$: If $F \in \mathbb{D}^{\infty}$, then take $(f_\epsilon)_\epsilon = F$ for all $\epsilon \in (0,1]$. Thus $\mathbb{D}^{\infty}$ is a faithful subalgebra of $\mathcal{H}_s$.

B) $\mathbb{D}^{-\infty} \subset \mathcal{H}_s$ (inclusion of Meyer-Watanabe distributions)
Firstly consider the canonical $\mathbb{R}^d$-valued functional $W_1(\omega) \equiv \omega(1),\ (\omega \in \Omega.)$ Then

$$DW_{1,i}(t) = te_i \in H,\ \ D_hW_{1,i}(\omega) = \ h_i(1) \ \text{ and thus } W_1(t) \in \mathbb{D}^{\infty}(\mathbb{R}^d);\ h \in H,\ t \in [0,1] \tag{2.2.1}$$

$$(e_1, \cdots, e_d) \ \text{ is an orthonormal basis in } \ \mathbb{R}^d \text{ and } \ (DW_{1,i}, h)_H = h_i(1) = D_hW_{1,i} \tag{2.2.2}$$

so that (2.1.2) is satisfied.

Given $T \in \mathbb{D}^{-\infty}$ we define the following functional in the convolution sense

$$f^{\phi}_{T,\epsilon}(\omega) = \left\langle T(\tilde{\omega}),\ \frac{1}{\epsilon^d}\phi\left(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon}\right)\right\rangle,\ \epsilon \in (0,1],\ \omega,\tilde{\omega} \in \Omega;\ \phi \in \mathcal{D}(\mathbb{R}^d) \tag{2.2.3}$$

($< .\,,\, . >$ denotes the bilinear form on $\mathbb{D}^{\infty} \times \mathbb{D}^{-\infty}$) and for fixed ϵ, $\phi\left(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon}\right) \in \mathbb{D}^{\infty}$.

We show that , $f^{\phi}_{T,\epsilon} \in \chi$ and also it is moderate:

$$D_h f^{\phi}_{T,\epsilon}(\omega) = \lim_{\lambda\to 0}\frac{1}{\epsilon^d}\left\langle T(\tilde{\omega},\ \phi\left(\frac{\omega(1)+\lambda h(1)-\tilde{\omega}(1)}{\epsilon}\right) - \phi\left(\frac{\omega(1)-\tilde{\omega}}{\epsilon}\right)\right\rangle$$

$$\overset{(2.1.2)}{=} \frac{1}{\epsilon^d}\left\langle T(\tilde{\omega}),\ \left(D\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon}),\ h\right)_H\right\rangle \overset{(2.1.4),(2.2.1)}{=} \frac{1}{\epsilon^{d+1}}\left\langle T(\tilde{\omega}),\ \left(\sum_{i=1}^{d}\partial_i\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon})\ te_i,\ h\right)_H\right\rangle$$

$$= \frac{1}{\epsilon^{d+1}}\left\langle T(\tilde{\omega}),\sum_{i=1}^{d}\partial_i\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon})\ \int_0^1 \dot{h}_i(s)ds\right\rangle = \frac{1}{\epsilon^{d+1}}\int_0^1\sum_{i=1}^{d}\left\langle T(\tilde{\omega},\ \partial_i\,\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon})\right\rangle \dot{h}_i(s)ds$$

$$= \frac{1}{\epsilon^{d+1}}\left(\sum_{i=1}^{d} < T(\tilde{\omega}),\ \partial_i\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon}) > te_i,\ h\right)_H .$$

Hence by (2.1.2)

$$Df^{\phi}_{T,\epsilon}(\omega) = \frac{1}{\epsilon^{d+1}}\sum_{i=1}^{d}\left\langle T(\tilde{\omega}),\ \partial_i\,\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon})\right\rangle\ te_i \in L^p(H)\ , t \in (0,1] \tag{2.2.4}$$

(Note: (2.2.4) can be obtained informally from (2.2.3) by inserting the gradient in the right side of the bilinear form $< T(\tilde{\omega}),\ .\ >$, , however this operation is not in general legitimate since the gradient of the right side is H-valued, thus does not belong to the domain of $T \in \mathbb{D}^{-\infty}$).

For the second gradient we use (2.1.3) with $E \longleftrightarrow H$ and for any $\tilde{h} \in H,\ \tau \in (0,1]$

$$(D_h Df^{\phi}_{T,\epsilon},\ \tilde{h})_H = \frac{1}{\epsilon^{d+2}}\left(\sum_{i=1}^{d}\left\langle T(\tilde{\omega}),\ (\sum_{j=1}^{d}\partial^2_{i,j}\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon}),\tau e_j,h)_H\right\rangle te_i,\ \tilde{h}\right)$$

$$= \frac{1}{\epsilon^{d+2}}\left(\sum_{i=1}^{d}\left\langle T(\tilde{\omega}),\ \sum_{j=1}^{d}\partial^2_{i,j}\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon})\,(\tau e_j,h)_H\right\rangle te_i,\ \tilde{h}\right)_H$$

$$= \frac{1}{\epsilon^{d+2}}\sum_{i=1}^{d}\left\langle T(\tilde{\omega})\sum_{j=1}^{d}\partial^2_{i,j}\,\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon})\,(\tau e_j,h)_H\right\rangle (te_i,\tilde{h})_H$$

$$= \frac{1}{\epsilon^{d+2}}\sum_{i,j=1}^{d}\left\langle T(\tilde{\omega}),\partial^2_{i,j}\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon}\right\rangle(\tau e_j,h)_H\,(te_i,\tilde{h})_H$$

$$= \frac{1}{\epsilon^{d+2}}\sum_{i,j=1}^{d}\left\langle T(\tilde{\omega}),\ \partial^2_{i,j}\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon})\right\rangle\ (\tau e_j\otimes te_i,\ h\otimes\tilde{h})_{H\otimes H} = (D^2 f^{\phi}_{T,\epsilon},h\otimes\tilde{h}),\ \text{thus}$$

$$D^2 f^{\phi}_{T,\epsilon}(\omega) = \frac{1}{\epsilon^{d+2}}\sum_{i,j=1}^{d}\left\langle T(\tilde{\omega}),\ \partial^2_{i,j}\phi(\frac{\omega(1)-\tilde{\omega}(2)}{\epsilon})\right\rangle\tau e_j\otimes te_i \in L^p(H\otimes H),\ \tau\in(0,1] \tag{2.2.5}$$

By induction

$$D^k f^{\phi}_{T,\epsilon} = \frac{1}{d+k}\sum_{i_1,i_2,\cdots,i_k=1}^{k}\left\langle T(\tilde{\omega}),\partial_{i_1,i_2,\cdots,i_k}\,\phi(\frac{\omega(1)-\tilde{\omega}(1)}{\epsilon})\right\rangle\otimes_{r=1}^{k} t_r e_{i_r};\ \ t_r\in(0,1],\ (r=1,\cdots,k) \tag{2.2.6}$$

To verify the inductive hypothesis we should show by (2.1.3)

$$(D^{k+1} f^{\phi}_{T,\epsilon}\,,\, \otimes_{j=1}^{k+1} h_j)_{H^{\otimes k+1}} \,=\, (D_{h_{k+1}} f^{\phi}_{T,\epsilon},\, \otimes_{j=1}^{k} h_j)_{H^{\otimes k}};\ h_j \in H,\, (j = 1, \cdots, k+1) \tag{2.2.7}$$

(The r,h.s. of (2.2.7) with the abbreviated symbol $\phi(\ \ )$)

$$\frac{1}{\epsilon^{d+k+1}} \left(\sum_{i_1,\cdots,i_k}^{d} < T(\tilde{\omega}),\ \sum_{i_{k+1}=1}^{d} (\partial^{k+1}_{i_1,\cdots,i_{k+1}} \phi(\ \ )\ t_{k+1}\, e_{i_{k+1}},\, h_{k+1})_H\ > \otimes_{r=1}^{k} t_r\, e_{i_r},\, \otimes_{j=1}^{k} h_j \right)_{H^{\otimes k}}$$

$$= \frac{1}{\epsilon^{d+k+1}} \sum_{i_1,\cdots,i_{k+1}}^{d} < T(\tilde{\omega}),\ \partial^{k+1}_{i_1,\cdots,i_{k+1}}\ \phi(\ \ ) > (t_{k+1} e_{i_{k+1}},\, h_{k+1})_H\, (\otimes_{r=1}^{k} t_r e_{i_r},\, \otimes_{j=1}^{k} h_j\,)_{H^{\otimes k}}$$

$$= \frac{1}{\epsilon^{d+k+1}} \left(\sum_{i_1,\cdots,i_{k+1}}^{d} < T(\tilde{\epsilon}),\, \partial_{i_1,\cdots,i_{k+1}}\, \phi(\ \ ) \otimes_{r=1}^{k+1}\, t_r e_{i_r},\, \otimes_{j=1}^{k+1} h_j \right)_{H^{\otimes k+1}} = \text{l.h.s.}$$

With an easy estimation we find that $|D^k f^{\phi}_{T,\epsilon}|^p_{H^{\otimes k}} < \infty$ so that $D^k\, f^{\phi}_{T,\epsilon} \in L^p(H^{\otimes k}),\ k \in \mathbb{N}/\{0\}$ Then referring to (2.1.5) we have $\|f^{\phi}_{T,\epsilon}\|_{k,p} < \infty$ and as (k,p) are arbitrary , by (2.1.7) $(f^{\phi}_{T,\epsilon})_\epsilon \in \chi$, furthermore $\|f^{\phi}_{T,\epsilon}\|_{k,p} = O(\epsilon^{-k-d})$ hence $(f^{\phi}_{T,\epsilon})_\epsilon \in \mathcal{M}_s$. Its class in $\mathcal{H}_s$ is a representative of $T \in \mathbb{D}^{-\infty}$.

If $\phi^* \in \mathcal{D}(\mathbb{R}^d)$ is another Schwartz test function, $f^{\phi}_{T,\epsilon}$ and $f^{\phi^*}_{T,\epsilon}$ are ϵ-equivalent :
Recall that in our set-up the Donsker's delta function is given canonically by $\delta_x(\omega(1))$ [19] where δ_x is the Dirac delta function at $x \in \mathbb{R}^d$. Then $\lim_{\epsilon\to 0} \left(f^{\phi}_{T,\epsilon} - f^{\phi^*}_{T,\epsilon} \right) =< T(\tilde{\omega}), \lim_{\epsilon \to 0} \left(\phi(\frac{\omega(1) - \tilde{\omega}(1)}{\epsilon}) - \phi^*(\frac{\omega(1) - \tilde{\omega}(1)}{\epsilon}) \right) >=$
$< T(\tilde{\omega}),\, \delta_{\tilde{\omega}(1)} - \delta_{\tilde{\omega}(1)} >= 0$.

2.3 Sharp(uniform) Topology in $\mathcal{H}_s$

$\mathbb{D}^\infty$ is a countably normed space; for a strictly increasing set $\{p_m\}, (1 < p_m < \infty, m = 1, 2, \cdots)$ of numbers $\mathbb{D}^\infty = \bigcap_{k\in\mathbb{N}} \bigcap \mathbb{D}^{p_m}_k$ (due to the monotonicity of $\|\,.\,\|_{k,p}$ norms). Arrange the set of countable norms as

$$\|\,.\,\|_{1,p_1};\ \ \|\,.\,\|_{2,p_1}, \|\,.\,\|_{1,p_2};\ \ \|\,.\,\|_{3,p_1}, \|\,.\,\|_{2,p_2}, \|\,.\,\|_{1,p_3};\ \ \cdots\cdots ; \|\,.\,\|_{n,p_1}, \|\,.\,\|_{n-1,p_2}, \cdots\cdots, \|\,.\,,\|_{1,p_n}; \cdots$$

Then we define a new set of countable norms $\mu_n,\ (n = 1, 2, \cdots$ by

$$\mu_n = \left(\sum_{j=1}^{n} \|\,.\,\|^2_{n+1-j,p_j} \right)^{1/2} \tag{2.3.1}$$

which are increasing ; also $\{\mu_n\}_{n\in\mathbb{N}}$ and $\{\|\,.\,\|_{k,p}\}$ norm systems are equivalent and create the same topology.
Furthermore $(f_\epsilon)_\epsilon \in \mathcal{H}_s$ is moderate in $\{\|\,.\,\|_{k,p}\}$ norms $\Leftrightarrow$ it is moderate in $\{\mu_n\}$ norms.

For $(f_\epsilon)_\epsilon \in \mathcal{M}_s$ the n-valuation of f, denoted by $v_n(f)$ is $\sup_{b\in\mathbb{Z}}\{\,\mu_n(f_\epsilon) = O(\epsilon^b)\}$. A family δ_n of ultrametric pseudistances on $\mathcal{M}_s$ is defined by $\delta_n(f,g) = \exp(-v_n(f-g)),\ \forall f, g \in \mathcal{M}_s$.

$(f_\epsilon)_\epsilon$ lies in $\mathcal{J}_s \Longleftrightarrow \forall n \in \mathbb{N},\, v_n(f) = +\infty$ (or equivalently $\forall n \in \mathbb{N},\ \delta_n(f,, 0) = 0$).

These definitions transfer naturally to the quotient space $\mathcal{H}_s$. Valuations and pseudo-distances have properties well-known in analysis. The ultrametric uniform structure and the topology constructed are called sharp uniform structure and sharp topology respectively.
Theorem. The space $\mathcal{H}_s$ is complete for the sharp uniform structure.
Proof. (Based on [12], prop. 1.31 and [15] , Thm 3.1). Let $(F_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}_s$ and let $(f_\epsilon)_\epsilon\ (n \in \mathbb{N})$ be a sequence of representatives in $\mathcal{M}_s$. That means $\forall n,\, \delta_n(f_r, f_l) \longrightarrow 0$ as $r \to \infty,\, l \to 0$. Considering the definition of the valuations we can extract a subsequence $(f_{q_n,\epsilon})_\epsilon \in \mathcal{M}_s$ with strictly increasing sequence $q_n,\ n \in \mathbb{N}$ of integers having the following property :
$\exists$ a decreasing sequence $\epsilon_k \downarrow 0\ (\epsilon_k \leq \frac{1}{2^k})$ such that

$$\forall \epsilon \in (0, \epsilon_k]\quad \mu_k(f_{q_k,\epsilon} - f_{q_{k-1},\epsilon}) < \epsilon^k,\ \ (k \geq 1) \tag{2.3.2}$$

Define for $k \in \mathbb{N}/\{0\}$

$$g_{k,\epsilon} \,=\, \begin{cases} f_{q_k,\epsilon} - f_{q_{k-1},\epsilon} & \text{if } \epsilon \in (0, \epsilon_k) \\ 0 & \text{if } \epsilon \in [\epsilon_k, 1] \end{cases} \tag{2.3.3}$$

and finally define

$$f_\epsilon \doteq f_{q_0,\epsilon} + \sum_{k=1}^{\infty} g_{k,\epsilon} \tag{2.3.4}$$

f_ϵ is locally finite. For any $n \in \mathbb{N}/\{0\}$

$$\mu_n(f_\epsilon) \le \mu_n(f_{q_0,\epsilon}) + \sum_{k=1}^{\infty} \mu_n(g_{k,\epsilon}) \le \mu_n(f_{q_0,\epsilon}) + \sum_{k=1}^{n-1} \mu_n(g_{k,\epsilon}) + \sum_{k=n}^{\infty} \mu_k(g_{k,\epsilon})$$

$$\le \mu_n(f_{q_0,\epsilon}) + \sum_{k=1}^{n-1} \mu_n(g_{k,\epsilon}) + \frac{\epsilon^n}{1-\epsilon}$$

(since μ_n is increasing and by (2.3.2)). thus $(f_\epsilon)_\epsilon \in \mathcal{M}_s$

As $f_\epsilon = f_{q_0,\epsilon} + f_{q_1,\epsilon} - f_{q_0,\epsilon} + \cdots\cdots + f_{q_{m-1},\epsilon} - f_{q_{m-2},\epsilon} + f_{q_m,\epsilon} - f_{q_{m-1},\epsilon} + \cdots$

We have by cancellations and for any $m \ge 1$

$$f_\epsilon - f_{q_m,\epsilon} = \sum_{k=m}^{\infty} g_{k+1,\epsilon}\ ; \quad \mu_m(f_\epsilon - f_{q_m,\epsilon}) = \mu_m(\sum_{k=m}^{\infty} g_{k+1,\epsilon}) \le \sum_{k=m}^{\infty} \mu_{k+1}(g_{k+1,\epsilon}) \le \frac{\epsilon^{m+1}}{1-\epsilon}, \quad (\epsilon \in (0,\epsilon_m))$$

This shows that the subsequence (f_{q_n}) converges to $(f_\epsilon)_\epsilon \in \mathcal{M}_s$ in sharp topology.
Thus the subsequence (F_{q_n}) converges to $F \in \mathcal{H}_s$, $(F_{q_n}$ and F are the classes of $(f_{q_n,\epsilon})_\epsilon$ and $(f_\epsilon)_\epsilon$ in $\mathcal{H}_s$ respectively).
As $(F_n)_{n\in\mathbb{N}}$ is Cauchy with a convergent subsequence it converges to F. □
Note. $\mathcal{H}_s$ is also a $\bar{\mathbb{C}}$ topological module, $\bar{\mathbb{C}}$ being the ring of generalized complex numbers. But this result is not needed in the sequel.

3 Inclusion of Hida Distributions

3.1 Real Case

We consider the Gelfand triplet $E \subset H \subset E^*$; H is a real separable Hilbert space , E is a countably Hilbert nuclear space which is densely and continuously imbedded in H and E^* is the dual of E.
By Minlos theorem there is a unique Gaussian measure μ on $(E^*, \mathcal{B}(E^*))$. Thus $(E^*, \mathcal{B}(E^*), \mu; H)$ forms a canonical Gaussian probability space. We denote the inner product and norm in $H($ or $(H^{\otimes n})$ by $(\,.\,,\,.\,)$ and $|\,.\,|$ respectively. $L^2(E^*,\mu)$ is abbreviated by (L^2).
Using the second quantization operator we construct the Hida testing functional space (E) and its dual $(E)^*$, the Hida distributions.
On the other hand the Meyer-Watanabe functional space $\mathbb{D}^\infty$ and the distribution space $\mathbb{D}^{-\infty}$ can also be constructed on $(E^*, \mathcal{B}(E^*), \mu)$ as well, the C-M space being replaced by H.
The starting point of the weak differential calculus will be then

$$(DF, h)_H = \lim_{\lambda\to 0} \frac{F(x+\lambda h) - F(x)}{\lambda}, \quad x \in E^*, \ h \in H \tag{3.1.1}$$

For $f \in E$, consider the functional W_f , defined by

$$W_f(x) \doteq < x, f >, \quad (< \,.\,,\,.\, > \text{ denotes the bilinear form on } E^* \times E \text{ or on } E \times E^*) \tag{3.1.2}$$

By Minlos' theorem $\int_{E^*} e^{i<x,f>}\, d\mu(x) = e^{-\frac{1}{2}|f|^2}$, hence W_f is a Gaussian random variable on E^*.
The linear map $f \longrightarrow W_f \,:\, E \longrightarrow L^2(E^*,\mu)$ can be extended to a linear isometry from H into $L^2(E^*,\mu)$. Now

$$(DW_f, h)_H = \lim_{\lambda\to 0} \frac{< f, x+\lambda h > - < f, x >}{\lambda} = < f, h > = (f,h)_H \quad \text{yielding} \quad DW_f = f \in E \subset H \tag{3.1.3}$$

It is known that for $(\mathbb{D}^\infty, \mathbb{D}^{-\infty})$ constructed on $(E^*, \mathcal{B}(E^*), \mu)$ we have $(E) \subset \mathbb{D}^\infty$ and $\mathbb{D}^{-\infty} \subset (E)^*$ both inclusions being dense. For $T \in (E)^*$ the counterpart of (2.2.3) will be $(d=1)$

$$g^{\phi,f}_{T,\epsilon}(x) = \frac{1}{\epsilon} << T(\tilde{x}), \phi\left(\frac{W_f(x) - W_f(\tilde{x})}{\epsilon}\right) >> ; x, \tilde{x} \in E^*, \ \phi \in \mathcal{D}(\mathbb{R}), \ |f|_H = 1 \tag{3.1.4}$$

where $<< \,.\,,\,.\, >>$ is the canonical bilinear form on $((E)^*, (E))$.

Polynomial functionals in the variables $\{W_{f_j}\}(j=1,2,\cdots n)$ belong to $\mathbb{D}^\infty \cap (E)$ and are dense in both of them. The right member of $<< .\,, . >>$ in (3.1.4) can be regarded as the limit of Taylor polynomials in W_f and is in (E) due to its completeness as a countably Hilbertian nuclear space (in fact a smooth functional).
We show that $(g^{\phi,f}_{T,\epsilon})_\epsilon \in \chi$ and that it is also moderate. We have :

$$(Dg_\epsilon, h)_H = \frac{1}{\epsilon} << T(\tilde{x}), \left(D\phi\left(\frac{W_f(x)-W_f(\tilde{x})}{\epsilon}\right), h\right)_H >>= \frac{1}{\epsilon^2} << T(\tilde{x}), \left(\phi'\left(\frac{W_f(x)-W_f(\tilde{x})}{\epsilon}\right) f, h\right)_H >>$$

$$= \frac{1}{\epsilon^2} << T(\tilde{x}), \phi'\left(\frac{W_f(x)-W_f(\tilde{x})}{\epsilon}\right) >> (f,h)_H = \frac{1}{\epsilon^2}\left(<< T(\tilde{x}), \phi'\left(\frac{W_f(x)-W_f(\tilde{x})}{\epsilon}\right) >> f, h\right)_H$$

which yields $Dg_\epsilon = \frac{1}{\epsilon^2} << T(\tilde{x}), \left(\phi'(\frac{W_f(x)-W_f(\tilde{x})}{\epsilon}\right) f >>$.

For higher order gradients we again use (2.1.3), in some abbreviated notation :

$$(D_h D g_\epsilon, \tilde{h})_H = \frac{1}{\epsilon^3}\left(<< T(\tilde{x}), (\phi^{''}(\ \ ) f, h)_H >> f, \tilde{h}\right)_H = \frac{1}{\epsilon^3}\left(<< T(\tilde{x}), \phi^{''}(\ \ )(f,h)_H >> f, \tilde{h}\right)_H$$

$$= \frac{1}{\epsilon^3} << T(\tilde{x}), \phi^{''}(\ \ ) >> (f,h)_H (f,\tilde{h})_H = \frac{1}{\epsilon^3} << T(\tilde{x}, \phi^{''}(\ \ ) >> (f\otimes f), h\otimes\tilde{h})_{H\otimes H}$$

$$= \frac{1}{\epsilon^3}\left(<< T(\tilde{x}), \phi^{''}(\ \ ) >> (f\otimes f, h\otimes\tilde{h}\right)_{H\otimes H} = (D^2 g_\epsilon, h\otimes\tilde{h})_{H\otimes H}$$

giving $D^2 g^{\phi,f}_{T,\epsilon}(x) = \frac{1}{\epsilon^3} << T(\tilde{x}), ,\phi^{''}\left(\frac{W_f(x)-W_f(\tilde{x})}{\epsilon}\right) >> (f\otimes f)$.

By an induction similar to the one in Section 2.2 :

$$D^k g^{\phi,f}_{T,\epsilon}(x) = \frac{1}{\epsilon^{k+1}} << T(\tilde{x}), \phi^{(k)}\left(\frac{W_f(x)-W_f(\tilde{x})}{\epsilon}\right) >> f^{\otimes k} = \frac{1}{\epsilon^{k+1}} << T(\tilde{x}), \phi^{(k)}\left(\frac{< x, f > - < \tilde{x}, f >}{\epsilon}\right) >> f^{\otimes k} \quad (3.1.5)$$

As clearly $|D^k g^{\phi,f}_{T,\epsilon}|^p_{H^{\otimes k}} < \infty$, we have $D^k g^{\phi,f}_{T,\epsilon} \in L^p(H^{\otimes k})$. so that by (2.1.5) $(g^{\phi,f}_{T,\epsilon})_\epsilon \in \chi$ and furthermore $(g^{\phi,f}_{T,\epsilon})_\epsilon \in \mathcal{M}_s$ as $\|g^{\phi,f}_{T,\epsilon}\|_{k,p} = O(\epsilon^{-k-1})$, hence $(E)^* \in \mathcal{H}_s$.
Different representatives obtained by $\tilde{\phi} \in \mathcal{D}(\mathbb{R})$ and/or by $\tilde{f} \in E, |\tilde{f}|_H = 1$, are either ϵ-equivalent or law equivalent or a combination of them according to Definitions 2.2.4, 2.2.5.

3.2 Complex case

For any topological vector space K on $\mathbb{R}$, denote by $V_\mathbb{C}$ its complexification, i.e. $V_\mathbb{C} = V + i\,V$.
If V is a Hilbert space and $u_1 + i\,v_1$ and $u_2 + i\,v_2$, are in $V_\mathbb{C}$, then their inner product is , $(u_1+iv_1, u_2+iv_2)_{V_\mathbb{C}} = (u_1,u_2)+(v_1,v_2)+i[(u_2,v_1)-(u_1,v_2)]$ thus $|u_1+iv_1|^2_{V_\mathbb{C}} = |u_1|^2_V + |v_1|^2_V$.
Let $\chi_\mathbb{C}$ be the set of all functions $(0,1] \longrightarrow \mathbb{D}^\infty_\mathbb{C} = \mathbb{D}^\infty + i\,\mathbb{D}^\infty$. Definitions 2.2.1 to 2.2.5 are modified accordingly to define $\mathcal{H}_{\mathbb{C},s}$, $\epsilon-$ and law equivalence.
$\mathcal{H}_{\mathbb{C},s}$ is also an algebra. We consider complex-valued functionals on the measure space $(E^*, \mathcal{B}(E^*), \mu)$. Letting $E_\mathbb{C} \ni f,\ f = f_1 + if_2,\ (f_j \in E,\ j = 1.2)$
For $x \in E^*$, $W_f =< x, f_1 + if_2 >=< x, f_1 > +i < x, f_2 >$.
The real and imaginary parts of W_f are independent Gaussian random variables with characteristic functions $e^{-\frac{1}{2}|f_j|^2}$, $(j = 1,2)$. If we take $|f_1|^2 = |f_2|^2 = 1/2$, then $W_f \in \mathbb{C}N(0,1)$, i.e. a standard complex-valued Gaussian random variable.
The linear map $f \to W_f$ from $E_\mathbb{C}$ to $L^2_\mathbb{C}(E^*,\mu) = (L^2)_\mathbb{C}$ can be extended to a linear isometry $H_\mathbb{C} \to (L^2)_\mathbb{C}$.

Consider the formal composition $\delta(W_f(x) - W_f(\tilde{x}))$, $x, \tilde{x} \in E^*$, where δ is the Dirac delta function. Using the linearity of Dirac delta it takes the form $\delta(< x, f_1 > - < \tilde{x}, f_1 >) + i\,\delta(< x, f_2 > - < \tilde{x}, f_2 >)$. This can be approximateed by the delta nets as

$\frac{1}{\epsilon}\phi\left(\frac{< x, f_1 > - < \tilde{x}, f_1 >}{\epsilon}\right) + i\,\frac{1}{\epsilon}\phi\left(\frac{< x, f_2 > - < \tilde{x}, f_2 >}{\epsilon}\right),\ \phi \in \mathcal{D}(\mathbb{R})\ \ (\dagger)$
Its limit can be regarded as a complex linear combination of two functions of so-called Donsker's type, (see also the next section). Let $T \in (E)^*_\mathbb{C}$, based on $(\dagger)$ define

$$\Theta^{\phi,f}_{T,\epsilon}(x) = \frac{1}{\epsilon} << T(\tilde{x}), \phi\left(\frac{< x, f_1 > - < \tilde{x}, f_1 >}{\epsilon}\right) + i\,\phi\left(\frac{< x, f_2 > - < \tilde{x} f_2 >}{\epsilon}\right) >>,\ f = f_1 + i\,f_2,\ x, \tilde{x} \in E^* \quad (3.2.1)$$

where $<< .\,, . >>$ is the canonical conjugate bilinear form on $(E)^*_{\mathbb{C}} \times (E_{\mathbb{C}})$. The use of two tests functions ϕ will not bring in an essential difference.
The real and imaginary parts of the right side of $<< .\,, . >>$ are both in (E) according to Section 3.1. Therefore the right side belongs to $(E)_{\mathbb{C}}$. (For a complete characterization of $(E)_{\mathbb{C}}$ and $(E)^*_{\mathbb{C}}$ in terms of S-transforms and U_β functionals , c.f [19] and [21]).

The calculation of the gradients of the functional (3.2.1) is basically the same as in the previous section except obvious modifications for the complex case. Thus

$$D\Theta^{\phi,f}_{T,\epsilon}(x) = \frac{1}{\epsilon^2} << T(\tilde{x}), \phi'\left(\frac{< x, f_1 > - < \tilde{x}, f_1 >}{\epsilon}\right) >> f_1 + i\,\frac{1}{\epsilon^2} << T(\tilde{x}), \phi'\left(\frac{< x, f_2 > - < \tilde{x}, f_2 >}{\epsilon}\right) >> f_2 \tag{3.2.2}$$

and

$$D^k\Theta^{\phi,f}_{T,\epsilon}(x) = \frac{1}{\epsilon^{k+1}} << T(\tilde{x}), \phi^{(k)}\left(\frac{< x, f_1 > - < \tilde{x}, f_1 >}{\epsilon}\right) >> f_1^{\otimes k}$$
$$+ i\,\frac{1}{\epsilon^{k+1}} << T(\tilde{x}), \phi^{(k)}\left(\frac{< x, f_2 > - < \tilde{x}, f_2 >}{\epsilon}\right) >> f_2^{\otimes k} \tag{3.2.3}$$

In parallel to the evaluations of Section 3.1, the $\|\,.\,\|_{k,p}$ norms of the real and imaginary parts of the right-hand sides in this expression are finite. This then shows that for arbitrary $k \in \mathbb{N}$ and $1 < p < \infty$ we have $\|\Theta^{\phi,f}_{T,\epsilon}\|_{k,p} < \infty$. Thus $\Theta^{\phi,f}_{T,\epsilon} \in \chi_{\mathbb{C}}$ and it is also moderate and represents $T \in (E)^*_{\mathbb{C}}$ in $\mathcal{H}_{\mathbb{C},s}$. In this way $(E)^*_{\mathbb{C}} \subset \mathcal{H}_{\mathbb{C},s}$.

3.3 Application to the Feynman Integrand.

Recall that Donsker's delta function in white noise theory set-up is given by $\delta_a(W_f) \equiv \delta(W_f - a)$ $; W_f(x) =< x, f >,\ x \in E^*, f \in E, a \in \mathbb{R}$.
It is shown to be a Hida distribution, therefore included in $\mathcal{H}_s$ with a representative $R(x) = \frac{1}{\epsilon}\phi(\frac{W_f(x) - a}{\epsilon}),\ \phi \in \mathcal{D}(\mathbb{R})$. As in section 3.1 we obtain

$$DR(x) = \frac{1}{\epsilon^2}\phi'(\frac{< f, x > -a}{\epsilon})\,f, \cdots, D^k R(x) = \frac{1}{\epsilon^{k+1}}\phi^{(k)}(\frac{< f, x > -a}{\epsilon})\,f^{\otimes k} \tag{3.3.1}$$

The other representatives are ϵ-equivalent.
$\delta(B_t)$, where B_t is the standard Brownian motion starting from 0 is also a Donsker's delta function. In the representative we may select $|f| = 1, f \in E$ and use $W_{\sqrt{t}f}$.

$\mathcal{E}_f = \exp\{W_f - \frac{1}{2} < f, f >\}$ is the exponential functional and for $f \in E_{\mathbb{C}}$ it is known to be in $(E)_{\mathbb{C}}$.

For all $f \in E,$ and $\lambda > 0,\ G(f) = \int_{E^*} \mathcal{E}_f\, \mu^{(\lambda)}(dx) = \exp\{\frac{\lambda^2-1}{2}|f|^2\}$ is shown to be the S-transform of a Hida distribution $F(\lambda) = S^{-1}G,$ [19] .
For $\lambda \in \mathbb{C},\ G(f)$ is still the S-transform of some element $F \in (E)^*_{\mathbb{C}},$ i.e. $F(\lambda) = S^{-1}G$.
The following short review is from [19] , [22] :
(In this section the initial Gelfand triplet of Section 3.1 can be taken as $S(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset S^*(\mathbb{R}^d)$ where $S^*(\mathbb{R}^d)$ denotes the set of tempered distributions).
The Schrödinger equation

$$\frac{\partial \Psi}{\partial t} = i\,(\frac{\Delta}{2} - V)\Psi;\ \ \Psi(0, x) = f(x) \tag{3.3.2}$$

where Δ is the Laplace operator in $\mathbb{R}^d,\ V$ is a real Borel function on $\mathbb{R}^d,$ (the potential). From the point of white-noise analysis it is preferable to start with the heat equation

$$\frac{\partial u}{\partial t} = (\frac{\lambda}{2}\Delta - i\,V)u\,; u(0, x) = f(x),\ \ \text{where}\ \ \lambda > 0 \tag{3.3.3}$$

The solution of (3.3.3) is given by the Feynman-Kac formula as

$$u(t, x) = \mathbb{E}\left[f(\sqrt{\lambda}B_t + x)\exp -i\int_0^t V(\sqrt{\lambda}s + xds\right] \tag{3.3.4}$$

where B_t is the standard Brownian motion starting from 0. In term of the Hida distribution $F(\sqrt{\lambda})$ this can be rewritten as

$$u(t, x) =<< F(\sqrt{\lambda})\,f(B_t + x)\,\exp -i\int_0^t V(B_s + x)dx,\ 1 >> \tag{3.3.5}$$

Let $f(x) = \delta_y(x)$. Then by (3.3.4) the fundamental solution of (3.3.3) is obtained as

$$u^\lambda(t,x,y) =<< F(\sqrt{\lambda})\,\delta(B_t - y + z)\,\exp -i\int_0^t V(B_s+x)dx,\,1>> \tag{3.3.6}$$

Suppose u^λ has an analytic continuation in λ, (the conditions for validity of this analytic continuation are given at Corollary 4.4 [23]), then the fundamental solution of (3.3.2) can be asserted to be

$$\Psi^\lambda(t,x,y) =<<\ F(\sqrt{i})\,\delta(B_t - y + x)\,e^{-i\int_0^t V(B_s+x)ds},\,1>>;\ \ F(\sqrt{i}) \in (E)^*_{\mathbb{C}} \tag{3.3.7}$$

The two factors $F(\sqrt{i})$ and $\delta(B_t - y + z)$ are in $\mathcal{H}_s$ by Section 3.2 and the paragraphs above. Therefore it remains to discuss the meaning of the exponential factor. $V(B_s+x)$ is expressed via a Bochner integral as

$$V(B_s+x) = \int_{\mathbb{R}^d} V(x)\,\delta(B_s+x-z)\,dz \tag{3.3.8}$$

For $\Delta_n = \{(t_1,\cdots,t_n):\ 0<t_1<\cdots<t_n<t\}$

$$\begin{aligned}\exp\{-i\int_0^t V(B_s+x)\,ds\} &= \sum_{n=0}^\infty (-i)^n \int_{\Delta_n}\prod_{j=1}^n V(B_{t_j}+x)\,d^n t\\ &= \sum_{n=0}^\infty (-i)^n \int_{\Delta_n} dt^n \prod_{j=1}^n \int_{\mathbb{R}^d} V(z_j)\,\delta(B_{t_j}+x-z_j)\,dz_j \end{aligned}\tag{3.3.9}$$

so that the full Feynman integrand for the propagator becomes

$$\begin{aligned}\sum_{n=0}^\infty (-i)^n \int_{\Delta_n} d^n t \prod_{j=1}^a ren \int_{\mathbb{R}^d} V(z_j)\,\delta(B_{t_j}+x-z_j)\,dz_j\,F(\sqrt{i})\delta(B_t-y+x) &= \sum_{n=0}^\infty (-i)^n \int_{\Delta_n} d^n t \prod_{j=1}^n \int_{\mathbb{R}^d} V(z_j)dz_j\\ \times F(\sqrt{i})\,\delta(B_t-y+x)\prod_{j=1}^n \delta(B_{t_j}+x-z_j)\end{aligned}\tag{3.3.10}$$

In [22], [23], [24]considerable effort is spent to give a reasonable meaning to the factor $F(\sqrt{i})\,\delta(\ \ )\prod$ as a Hida distribution. But obviously this factor is in algebra $\mathcal{H}_{s,\mathbb{C}}$. furthermore in [21] it is shown that under the assumptions that $d=1$, and $V(y)dy$ is a compactly supported signed measure, the product of the three terms in the right-hand side of (3.3.5) is a Hida distribution. To perform the t_i-integrations in (3.3.9) and the convergence of the series in the first factor of (3.3.9) the assumption $d=1$ is essential. For in the evaluation the integral $M_n = \int_{\Delta_n} d^n t \prod |t_i - t_{i-1}|^{-\frac{d}{2}}$ is needed and it exists only for $d=1$. In this case M_n rapidly decreases and $\sum_{n=1}^\infty M_n < \infty$.
However in our case when $d=1$, without needing the introduction of Δ_n and the elaborate evaluations of (3.3.8) and (3.3.9), we can show the following:
Theorem. If $d=1$ and $V(z)\,dz$ is a bounded signed measure, then the product of three terms in (3.3.6) is in $\mathcal{H}_{s,\mathbb{C}}$.

Proof. Firstly let us show that $V(B_s+x) = \int_{\mathbb{R}} V(z)\,\delta(B_s+x-z)\,dz$ given by (3.3.7) is in $\mathcal{H}_s$. It will have a representative

$$\Gamma_\epsilon^{\phi,f}(\psi) = \frac{1}{\epsilon}\int_{\mathbb{R}} V(z)\,\phi\left(\frac{(W_{\sqrt{s}f}(\psi) - (z-x)}{\epsilon}\right)dz,\ \ f\in E, |f|=1, \phi\in\mathcal{D}(\mathbb{R}),\ \psi\in E^* \tag{3.3.11}$$

By bounded convergence theorem the operators D_h and D, can be inserted into the integral and an approach parallel to those in sections 2.2 and 3.1 yields

$$D\Gamma_\epsilon^{\phi,f}(\psi) = \frac{1}{\epsilon^2}[\int_{\mathbb{R}}\phi'\left(\frac{\sqrt{s}<\psi,x> -(z-x)}{\epsilon}\right)\sqrt{s}\,V(z)\,dz]\,f$$

and

$$D^k\Gamma_\epsilon^{\phi,f}(\psi) = \frac{1}{\epsilon^{x+1}}[\int_{\mathbb{R}}\phi^{(k)}\left(\frac{\sqrt{s}<\psi,f> -(z-x)}{\epsilon}\right)s^{k/2}\,V(z)dz]\,f^{\otimes k} \tag{3.3.12}$$

$$|D^k\Gamma_\epsilon^{\phi,f}|^p_{H^{\otimes k}} = \frac{1}{\epsilon^{k+1}}|\int_{supp\ \phi^{(k)}}\phi^{(k)}\left(\frac{\sqrt{s}<\psi,f> -(z-x)}{\epsilon}\right)s^{k/2}V(z)dz|^p < \infty \tag{3.3.13}$$

since $|f^{\otimes k}|_{H^{\otimes k}} = 1$. As $V(z)dz = V(z)^+dz - V(z)^-dz$ is a bounded signed measure (3.3.12) shows that $\|D^k\Gamma_\epsilon^{\phi,f}\|_{L^p(\mu;H^{\otimes k})} < \infty$ As $k\in\mathbb{N}$ and $1<p<\infty$ are arbitrary it follows from (2.1.5) that $\Gamma_\epsilon^{\phi,f}\in\mathbb{D}^\infty$, $(0<\epsilon<1)$, furthermore $\|\Gamma_\epsilon^{\phi,f}\|_{k,p} = O(\epsilon^{-k-1})$ hence $(\Gamma_\epsilon^{\phi,f})_\epsilon\in\mathcal{M}_s$, then its class in $\mathcal{H}_s$ represents $V(B_s+x)$.

$\int_0^t V(B_s+x)\,ds$ regarded as the limit of finite sums over the partitions of $[0,1]$ is in $\mathcal{H}_s$ due to its completion.
Now $e^{-i\int_0^t V(B_s+x)\,ds} = \cos\int_0^t V(B_s+x)ds - i\sin\int_0^t V(B_s+x)$, and as a complex linear combination of smooth, bounded functions of members of $\mathcal{M}_s$, (2.1.4) this expression is included in $\mathcal{M}_{s,\mathbb{C}}$. Its class in $\mathcal{H}_{s,\mathbb{C}}$ represents $e^{-i\int_0^t V(B_s+x)ds}$. As a consequence each factor of the product in (3.3.6) is in $\mathcal{H}_{s,\mathbb{C}}$, so is their product. □

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Concluding Remarks.
1) Present approach may have potentiality in studying the generalized solutions of non-linear stochastic partial differential equations as in [11].
2) In this paper non-linearities of polynomial order of $1/\epsilon$ was considered. For non-linearities growing faster than polynomial order (e.g. of exponential order) an asympotic scale $a_n(\epsilon)$ accompanied by a set of postulates can be introduced, (c.f. [16]). In our case $a_n(\epsilon) = \epsilon^n$.

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