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Method for solving notorious calculus problems speeds particle physics computations

Numerical technique reduces vexing Feynman integrals to simpler linear algebra

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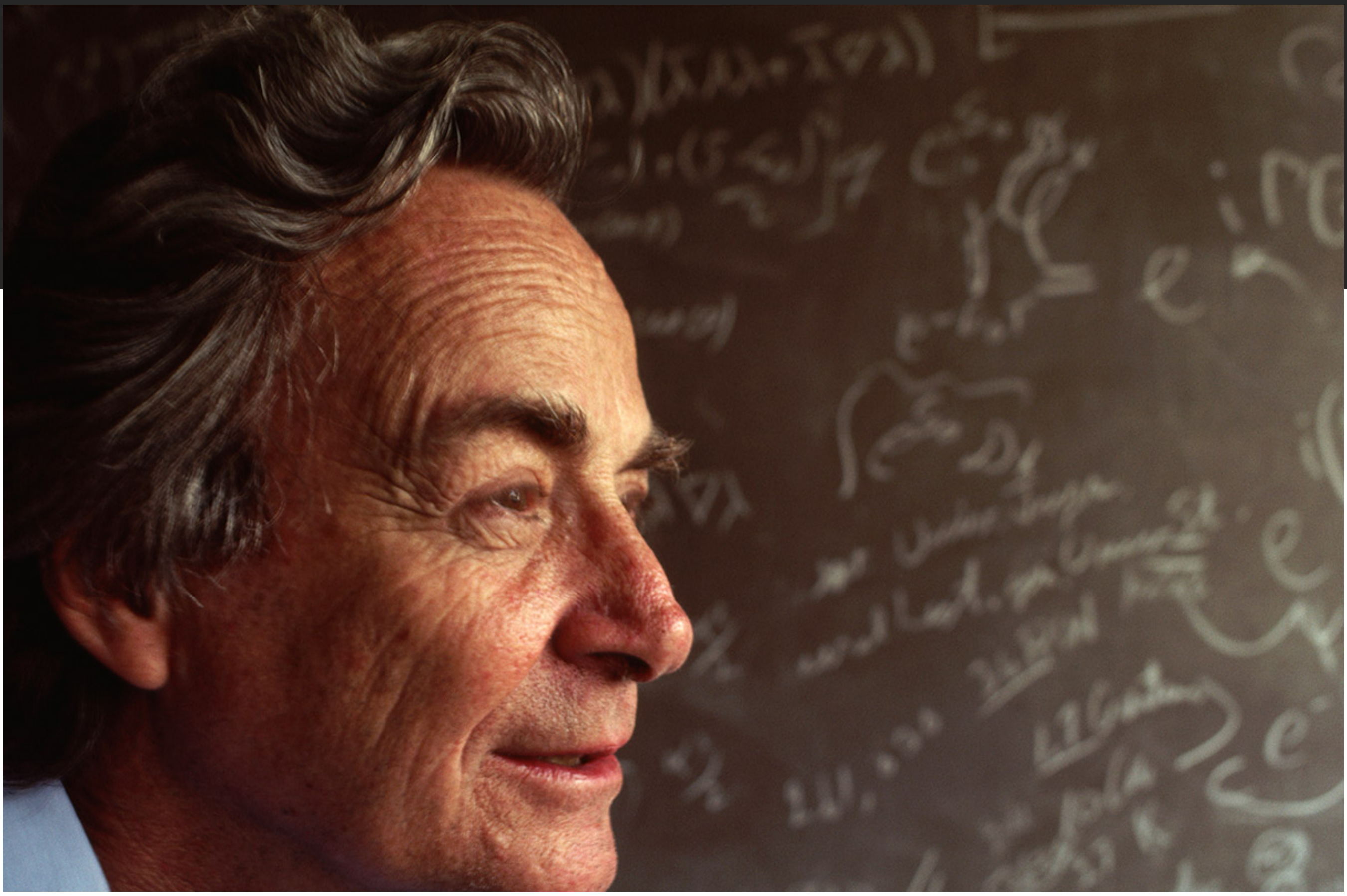


Richard Feynman invented stick figures to help sort through particle physics calculations—but each is associated with a difficult integral. KEVIN FLEMING/CORBIS VIA GETTY IMAGES

For decades, theoretical particle physicists have struggled with vexing calculus problems called Feynman integrals. They are central to every calculation they make—from predicting how magnetic a particle called the muon should be, to estimating the rate at which Higgs bosons should emerge at the Large Hadron Collider (LHC). Now, theorists have found a way to solve the integrals numerically by reducing them to linear algebra. The method promises faster and more precise theoretical calculations, which are essential for searching for hints of new particles and forces.

“Sometimes people come up with some deep mathematical insights into these Feynman integrals, but they actually don’t help you to calculate things,” says Ayres Freitas, a theoretical physicist at the University of Pittsburgh. “This method will help.”

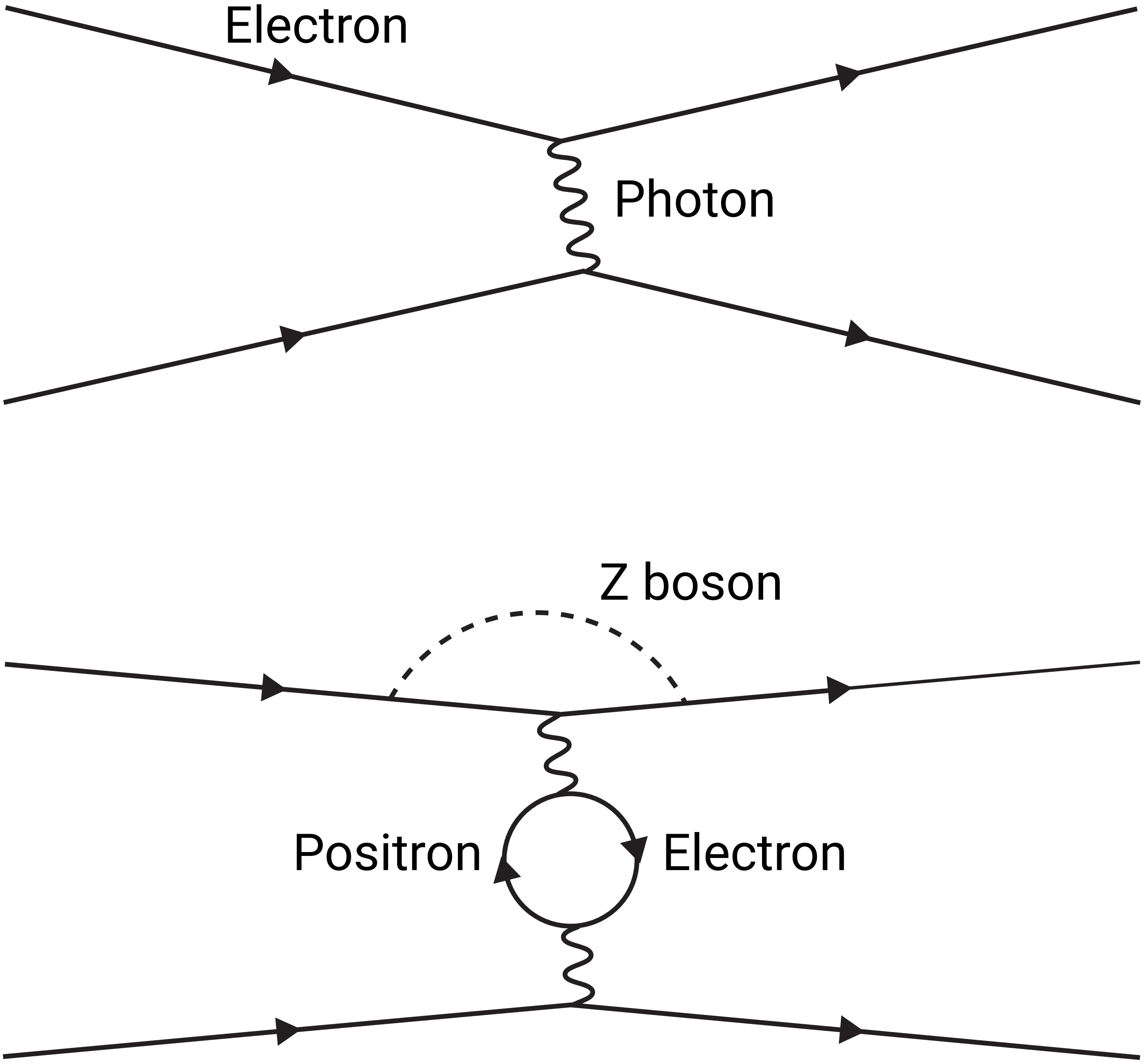
“It’s surprising that [the method] works so well,” says Stefan Weinzierl, a theoretical physicist at Johannes Gutenberg University of Mainz who has written an 800-page book on the integrals. “In principle, it’s absolutely general, so you can treat any Feynman integral with it.”

Feynman integrals have plagued particle theorists since the rise of quantum field theory in the mid–20th century. Each integral corresponds to one of the quirky diagrams concocted in 1948 by Richard Feynman to quickly figure out what to calculate for a particular particle interaction. For example, one electron can deflect another when the two exchange a photon, so the simplest Feynman diagram for the process consists of two lines representing the electrons and a wavy line connecting them that represents the photon. The doodle serves a serious purpose: Each line stands for a component in an integral yielding the probability of interaction given each electron’s initial and final momentum.

Even a graduate student can handle that integral. However, electron interactions are much more complex. For example, the exchanged photon can morph into an electron-positron pair and back into a photon, or one electron can emit and reabsorb a particle called a Z boson. Such “perturbations” correspond to Feynman diagrams with one or more closed internal loops, and precise calculations must account for at least the simpler loop diagrams.

The integrals for loop diagrams are pathological, however. With multiple variables, many are impossible to do by hand and require numerical methods. Even then, the integrals tend to blow up, forcing theorists to employ myriad tricks and techniques to make the problems tractable. In most cases, theorists struggle to get past two-loop integrals, Freitas says, of which any particular interaction may have thousands.

That’s no mere academic issue, notes Johannes Henn, a theoretical physicist at the Max Planck Institute for Physics. Every 2 years, particle physicists meet at a resort town in France to compile a wish list of improved theoretical calculations they want to compare with experimental results from the LHC as they search for holes in physicists’ prevailing theory of fundamental particles, the standard model. In striving to fulfill those wishes, Henn says, “the bottleneck is our ability to actually evaluate Feynman diagrams and integrals at the loop level.”



The simplest Feynman diagram for one electron deflecting another (top) and a more complicated two-loop diagram, which stands for a much more difficult integral C. BICKEL/*SCIENCE*

Now, Zhi-Feng Liu and Yan-Qing Ma, theoretical physicists at Peking University, have developed a method that reduces any such integral to a more tractable problem in linear algebra, they reported last week in *Physical Review Letters*. Liu and Ma begin with an old observation that any Feynman integral with a given number of loops can be written as a linear combination of certain other "master" integrals with the same number of loops. (A linear combination is a sum with coefficients: A Manhattan cocktail is a linear combination of two parts whisky and one part vermouth.) So, for a given number of loops, if theorists can solve the master integrals, they can solve any Feynman integral.

Liu and Ma then invoke a second old theorem that allows them to write the derivative of each master integral as a linear combination of the same master integrals. That produces a set of differential equations relating all of the master integrals. At that point, instead of crunching the integrals directly, theorists can deduce them implicitly by solving the differential equations numerically. That's something a computer can do easily, Ma says.

There's a catch, however. To solve the differential equations, theorists need some initial values or "boundary conditions" for the master integrals. That typically requires finding some symmetry that makes the boundary conditions simple or biting the bullet and doing the integration for some fixed values of the input variables. Here, however, Liu and Ma resort to a breakthrough Ma's team made in 2017. They modify the master integrals by inserting a new parameter and then performing the derivatives with regard to it. By setting that extra parameter to infinity, they get a set of differential equations for which determining the boundary conditions is much easier. Having done that, they adjust the new parameter back to zero to regain the differential equations for the master integrals, complete with the needed boundary conditions.

Here's how Liu and Ma determine the boundary condition: With their new parameter set to infinity, that task involves evaluating integrals for "vacuum diagrams" that have no external legs and only loops. But each of those can be equated to a related diagram with one loop fewer. Iterating that move eventually gives the vacuum integrals as linear combinations of integrals with no loops, which are trivial. Thus, the boundary condition problem reduces to pure linear algebra.

Using results from the literature, Liu and Ma confirmed their technique reproduces correct results for five-loop integrals. In January, the team made the technique available via a downloadable software package. In recent months, Ma claims, 80%

of papers posted to the arXiv preprint server involving the computation of Feynman integrals have used the package.

Of course, theorists don’t get something for nothing. The technique sidesteps brute-force numerical integration, but it requires a lot more algebra. “The main computational time in our method is not to solve the differential equations, but to get the differential equations,” Ma says. Some computations could require a computer to solve a half-billion linear equations, he says.

Still, the speedy new technique is likely to prove widely useful, Weinzierl says, even if it is largely a clever combination of earlier results. “Ingredients like flour, salt, sugar, and so on are all readily available,” he says, “but if a cook makes a very delicious meal out of them, you appreciate it.”

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