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ABSTRACT BOOK

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“Functional Analysis in Interdisciplinary Applications”

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of the conference FAIA2023

Edited by

Allaberen Ashyralyyev,

Michael Ruzhansky,

Makhmud Sadybekov

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Bahçeşehir University (Türkiye), Analysis & PDE Center, Ghent University (Belgium), and Institute of Mathematics and Mathematical Modeling (Kazakhstan), being the organizers of the conference, present this Abstract book, which contains brief abstracts of the reports of the participants of the International Mathematical Conference "Functional Analysis in Interdisciplinary Applications" (FAIA2023).
This Conference aims to bring together mathematicians working in the functional analysis and its applications.

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FOREWORD

On behalf of the Organizing Committee, we are pleased to invite you to the International Mathematical Conference "Functional Analysis in Interdisciplinary Applications" (FAIA2023).

This onference aims to bring together mathematicians working in the functional analysis and its applications.

The meeting will be held on October 02–October 07, 2023 in Antalya, Türkiye. The conference will consist of plenary lectures, and sectional oral presentations.

This Astract book contains brief abstracts of the reports of the participants of the FAIA2023. The collection of abstracts is organized in alphabetical order by the last name of the first author.

We would like to thank our main sponsors Bahçeşehir University, Türkiye, Institute of Mathematics and Mathematical Modeling, Kazakhstan, and Ghent Analysis & PDE Center, Belgium. We also would like to thank to all participants and Technical Program Committee Members.

With our best wishes and warm regards,

Prof. Allaberen Ashyralyev

Prof. Michael Ruzhansky

Prof. Makhmud Sadybekov

Chairs of FAIA2023

Contents

Boundedness and Compactness of Commutator Bilinear Riesz Potential in Generalized Morrey Spaces

Aidos ADILKHANOV[1,c], Talgat AKHAZHANOV[1,b] Dauren MATIN[1,a]

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In this paper, we obtain that $b \in VMO(\mathbb{R}^n)$ the commutator $[b; I_\alpha]$ is compactness in generalized Morrey space from $M_{p_1}^{w_1} \times M_{p_2}^{w_2}$ to M_q^w., for some appropriate indices $p_1; p_2; q; w; w_1; w_2; \alpha$.

The main goal of this paper is to find sufficient conditions for the commutator of bilinear $[b, I_\alpha]$ to be precompact in the generalized Morrey space $M_p^w(\mathbb{R}^n)$.

Let $1 \le p \le \infty$, w be a measurable non-negative function on $(0, \infty)$ that is not equivalent to zero. The generalized Morrey space $M_p^{w(\cdot)} \equiv M_p^{w(\cdot)}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite norm $\|f\|_{M_p^{w(\cdot)}} \equiv \sup\limits_{x \in \mathbb{R}^n} \sup\limits_{r>0} \left(w(r) \|f\|_{L_p(B(x,r))} \right) < \infty$

For $0 < \alpha < 2n$, the bilinear fractional integral is defined by $I_\alpha(f,g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}} dydz$

For a function $b \in L_{loc}(\mathbb{R}^n)$ by M_b denote multiplier operator $M_b f = bf$, where f is measurable function. Then the commutator between I_α and M_b is defined by $[b, I_\alpha]_1 (f,g)(x) = M_b I_\alpha - I_\alpha M_b = \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{(b(y)-b(x))f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}} dydz$,

$[b, I_\alpha]_2 (f,g)(x) = M_b I_\alpha - I_\alpha M_b = \int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{(b(z)-b(x))f(y)g(z)}{(|x-y|+|x-z|)^{2n-\alpha}} dydz$

It is said that the function $b(x) \in L_\infty(\mathbb{R}^n)$ belongs to the space $BMO(\mathbb{R}^n)$, if $\|b\|_* = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int\limits_Q |b(x) - b_Q| \, dx = \sup_{Q \in \mathbb{R}^n} M(b, Q) < \infty$, where Q - cube $\mathbb{R}^n$ and $b_Q = \frac{1}{|Q|} \int\limits_{\mathbb{R}^n} f(y) dy$.

By $VMO(\mathbb{R}^n)$ we denote the BMO-closure $C_0^\infty(\mathbb{R}^n)$, where $C_0^\infty(\mathbb{R}^n)$ the set of all functions from $C^\infty(\mathbb{R}^n)$ with compact support. Through the $\chi(A)$ denotes the characteristic function of the set $B \subset \mathbb{R}^n$, and ${}^cA$ denotes the complement of A.

The main goal of this paper is to find sufficient conditions for the commutator of bilinear $[b, I_\alpha]$ to be precompact in the generalized Morrey space $M_p^w(\mathbb{R}^n)$.

Let $1 \le p \le \infty$, w be a measurable non-negative function on $(0, \infty)$ that is not equivalent to zero. The generalized Morrey space $M_p^{w(\cdot)} \equiv M_p^{w(\cdot)}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_p^{w(\cdot)}} \equiv \sup_{x \in \mathbb{R}^n} \sup_{r>0} \left(w(r) \|f\|_{L_p(B(x,r))} \right) < \infty$$

Theorem 1. *[1] Let $1 \leq p \leq \infty$ and $w \in \Omega_{p\infty}$. Let's assume that the set $S \subset M_p^{w(\cdot)}$ satisfies the following conditions:*

$$\sup_{f\in S} \|f\|_{M_p^{w(\cdot)}} < \infty,$$

$$\lim_{|u|\to 0} \sup_{f\in S} \|f(\cdot + u) - f(\cdot)\|_{M_p^{w(\cdot)}} = 0,$$

$$\lim_{r\to\infty} \sup_{f\in S} \left\|f\chi_{{}^cB(0,r)}\right\|_{M_p^{w(\cdot)}} = 0.$$

Then S is a pre compact set in $M_p^{w(\cdot)}$.

Suppose that the continuous increasing functions w_1, w_2 in $[0;\infty)$ satisfy the following conditions $(j = 1, 2)$:

a) $w_j(0) = 0$;

b) $lim_{r\to\infty} w_j(r) < \infty$;

c) There exists a constant D, satisfying $1 \leq D < 2^n$, such that $w_j(2r) \leq Dw_j(r)$ for any $r > 0$;

d)$w(r)^{\frac{1}{p}} = w_1(r)^{\frac{1}{p_1}} w_2(r)^{\frac{1}{p_2}}$

Theorem 2. *Let $0 < \alpha < 2n$, $0 < \lambda < n$, $\frac{1}{2} < p < \frac{n-\lambda}{\alpha}$, $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, functions $w, w_1, w_2 \in \Omega_{p,\infty}$ satisfying conditions (a)-(d) for any $r > 0$. Then I_α is bounded bilinear operator from $M_{p_1}^{w_1} \times M_{p_2}^{w_2}$ to M_q^w. For $b \in BMO(\mathbb{R}^n)$, the commutator $[b, I_\alpha]_i$ is also bounded from $M_{p_1}^{w_1} \times M_{p_2}^{w_2}$ to M_q^w, $i = 1, 2$,*

$$\| [b, I_\alpha]_i (f, g)\|_{M_q^w} \leq C\|b\|_{BMO}\|f\|_{M_{p_1}^{w_1}} \|f\|_{M_{p_2}^{w_2}} \tag{1}$$

Theorem 3. *Let $0 < \alpha < 2n$, $0 < \lambda < n$, $\frac{1}{2} < p < \frac{n-\lambda}{\alpha}$, $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $1 < q < \infty$,$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, function $w, w_1, w_2 \in \Omega_{p,\infty}$ satisfying conditions (a)-(d) for any $r > 0$ and $b \in VMO(\mathbb{R}^n)$, (1). Then commutator $[b, I_\alpha]_i$ is a compact from $M_{p_1}^{w_1} \times M_{p_2}^{w_2}$ to M_q^w, $i = 1, 2$.*

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Keywords: commutator, single integral operator, compactness, global Morrey-type space.

2020 Mathematics Subject Classification: 35Q79, 35K05, 35K20

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Solution of the Cauchy Problem of the Dynamics of a Thermoelastic Rod Using the Vladimirov Method

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Nonstationary boundary value problems of the dynamics of thermoelastic rods are considered. Based on the generalized function method, a solution to the Cauchy problem was constructed using a model of uncoupled thermoelasticity under the influence of non-stationary force and thermal loads of various types. Regular integral representations of the solution to the Cauchy problem for a thermoelastic rod are obtained.

1. Statement of the Cauchy problem for the equations of uncoupled thermoelasticity. We consider a thermoelastic rod, the equations of state of which have the form [1]:

$$\begin{cases} \rho c^2 u_{,xx} - \rho u_{,tt} - \gamma\theta_{,x} + \rho F_1(x,t) = 0 \\ \theta_{,xx} - k^{-1}\theta_{,t} + F_2(x,t) = 0 \end{cases} \tag{2}$$

where $x \in R^1$, $t \geq 0$. $F_1(x,t)$ -longitudinal component of external force per unit length, $F_2(x,t)$ - quantity characterizing the power of the heat source.

Thermoelastic stresses in the rod are related to displacement by the Duhamel-Neumann law:

$$\sigma(x,t) = \rho c^2 u_{,x} - \gamma\theta \tag{3}$$

The initial conditions are known:

$$\begin{gathered} u(x,0) = u_0(x),\ u_{,t}(x,0) = \nu_0(x),\ \theta(x,0) = \theta_0(x), \\ u_0(x) \in C^2(R^1),\ \theta_0(x) \in C^2(R^1),\ \nu_0(x) \in C^2(R^1). \end{gathered} \tag{4}$$

$C^n(R^1)$ where is the space of functions differentiable up to the nth order on R^1.

It is required to find solutions to equations (1) with initial conditions (3), which satisfy the radiation conditions: under $u(x,t) \to 0$, $\theta(x,t) \to 0$, under $|x| \to \infty$, $\forall t$ the action of arbitrary forces and heat sources: $F_j(x,t) = L_1(R^1)$, under $j = 1,2$.

2. Statement of the Cauchy problem in the space of generalized functions. To solve the problem we use the method developed by V.S. Vladimirov. to solve the Cauchy problem of wave equations [2]. Let us introduce the following regular generalized functions:

$$\hat{u}(x,t) = u(x,t)H(t),\ \hat{\theta}(x,t) = \theta(x,t)H(t),\ \hat{F}(x,t) = F(x,t)H(t) \tag{5}$$

System (1) in the space of generalized functions will have the following form:

$$\begin{gathered} c^2\hat{u}_{,xx} - \hat{u}_{,tt} - \tilde{\gamma}\hat{\theta}_{,x} = -F_1(x,t)H(t) - \nu_0(x)\delta(t) - u_0(x)\delta'(t) = -\hat{F}_1(x,t) \\ \frac{\partial\hat{\theta}}{\partial x^2} - k^{-1}\frac{\partial\hat{\theta}}{\partial t} = -F_2(x,t)H(t) + k^{-1}\theta_0(x)\delta(t) = -\hat{F}_2(x,t). \end{gathered} \tag{6}$$

where $\tilde{\gamma} = \frac{\gamma}{\rho}$. The right-hand sides of these equations (5) include the initial conditions as singular mass forces and heat sources:

$$\begin{gathered} \hat{F}_1(x,t) = F_1(x,t)H(t) + \nu_0(x)\delta(t) + u_0(x)\delta'(t), \\ \hat{F}_2(x,t) = F_2(x,t)H(t) + k^{-1}\theta_0(x)\delta(t). \end{gathered} \tag{7}$$

The solution to this problem in the space of generalized functions has the form of tensor-functional convolution [1]:

$$\begin{aligned} \hat{u}(x,t) &= u(x,t)H(t) = \hat{U}_1^k(x,t) * \hat{F}_{\triangle k}(x,t),\ (k=1,2), \\ \hat{\theta}(x,t) &= \theta(x,t)H(t) = \hat{U}_2^k(x,t) * \hat{F}_{\triangle k}(x,t),\ (k=1,2). \end{aligned} \tag{8}$$

3. Green's tensor of the equations of unbound thermoelasticity. Green's tensor $U_i^j(x,t)$ is a matrix of fundamental solutions of system (1) under the action of pulsed concentrated forces and a heat source of the form:

$$F_1 = \delta(x)\delta(t)\delta_1^j,\ F_2 = \delta(x)\delta(t)\delta_2^j,\ j=1,2 \tag{9}$$

The Green's tensor is constructed and has the following form [3]:

$$\begin{aligned} U_1^j(x,t) &= \delta_1^j k^{-1}\frac{\partial \Sigma_1}{\partial t} - \delta_1^j \frac{\partial \Sigma_1}{\partial x^2} - \delta_2^j \tilde{\gamma}\frac{\partial \Sigma_1}{\partial x} - \delta_1^j \Sigma_3(t)\delta(x) + \delta_2^j \tilde{\gamma}\frac{\partial \Sigma_2}{\partial x}, \\ U_2^1 &= 0,\ U_2^2 = c^2\Sigma_3(x,t) + c^2\frac{\partial^2 \Sigma_2}{\partial x^2} - \frac{\partial^2 \Sigma_2}{\partial t^2}, ...j=1,2. \end{aligned} \tag{10}$$

Knowing the Green's tensor, it is possible to construct a solution to system (6) for any sources in the form of a tensor-functional convolution (7).

4. Solutions to the Cauchy problem. To obtain an integral representation of the generalized solution, we take convolutions (7) taking into account (6), using the properties of the δ- function and its derivative:

$$\begin{aligned} \hat{u}(x,t) = u(x,t)H(t) = \hat{U}_1^1(x,t) * F_1(x,t)H(t) + \hat{U}_1^1(x,t) \underset{x}{*} \nu_0(x) + \\ \frac{\partial}{\partial t}\hat{U}_1^1(x,t) \underset{x}{*} u_0(x) + \hat{U}_1^2(x,t) \underset{x}{*} F_2(x)H(t) + \hat{U}_1^2(x,t) \underset{x}{*} k^{-1}\theta_0(x) \end{aligned} \tag{11}$$

$$\hat{\theta}(x,t) = \theta(x,t)H(t) = \hat{U}_2^2(x,t) * F_2(x,t)H(t) + \hat{U}_2^2(x,t) \underset{x}{*} k^{-1}\theta_0(x) \tag{12}$$

I would like to note that all the initial conditions of the Cauchy Problem are included in the right side of relations (10), (11).

So, using the apparatus of generalized function theories, we solved the Cauchy problem. The solutions obtained make it possible to find the stress-strain state in the rod under the action of various common power and heat sources using formulas and theorems, i.e. if we pose the Cauchy Problem under new initial conditions (3). We substitute these data into the right side of the resulting formulas (10) and (11) and obtain new solutions to the Cauchy Problem.

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Keywords: Cauchy problem, rod, uncoupled thermoelasticity, generalized function method, heat flow, temperature, displacement and stress.

2020 Mathematics Subject Classification: 74A15, 65R20, 35L05, 35K05.

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Approximation of Functions of Two Variables of Bounded P-Fluctuation by Polynomials with Respect to Walsh Systems

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IN THIS PAPER, WE PROVE DIRECT AND CONVERSE THEOREMS OF APPROXIMATION OF FUNCTIONS OF TWO VARIABLES OF BOUNDED P-FLUCTUATIONS BY WALSH POLYNOMIALS IN THE NORM OF THE SPACE.

Let $1 \le p < \infty$ and the function $f(x, y)$ be defined on the set $[0, 1)^2$ and for any set I denote by $I_{j_1,j_2}^{(n_1,n_2)}$ rectangular $\left[\frac{j_1-1}{2^{n_1}}, \frac{j_1}{2^{n_1}}\right) \times \left[\frac{j_2-1}{2^{n_2}}, \frac{j_2}{2^{n_2}}\right)$. Denote by $osc\left(f, [a, b)^2\right) = \sup\limits_{(x,y),(x',y')\in I} |f(x, y) - f(x', y')|$.

By definition, $\kappa_p(f, n_1, n_2) := \left(\sum_{j_1=1}^{2^{n_1}} \sum_{j_2=1}^{2^{n_2}} \left(osc\left(f, I_{j_1,j_2}^{(n_1,n_2)}\right)\right)^p\right)^{1/p}$. If $V_p(f) := \sup\limits_{\substack{n_1\in P \\ n_2\in P}} \kappa_p(f, n_1, n_2) < \infty$, then $f(x, y)$ is called the function of bounded p-fluctuation. In one variable case, the definition was introduced by Onneweer and Waterman [1] Now we introduce a discrete modulus of continuity $V_p(f)_{n_1,n_2} = \sup\limits_{\substack{k_1\ge n_1 \\ k_2\ge n_1}} \kappa_p(f, k_1, k_2)$. The set of functions $f(x, y)$, for which $V_p(f) < \infty$, we denote by $FV_p\,[0, 1)^2$ $(1 \le p < \infty)$. Let us introduce one more discrete group modulus of continuity related to the space $FC_p\,[0, 1)^2$ $(1 < p < \infty)$, by the formula

$$V_p(f)^*_{n_1,n_2} = \sup_{\substack{0\le h_1<\frac{1}{2^{n_1}} \\ 0\le h_2<\frac{1}{2^{n_2}}}} \|f(x_1 \oplus h_1, x_2 \oplus h_2) - f(x_1, x_2)\|_{p,F}\,.$$

Theorem 1. *Let* $1 < p < \infty$, $n, m \in N$, $f \in FV_p\,[0, 1)^2$. *Then the following inequality is valid:* $\frac{1}{2}V_p(f)^*_{m,n} \le E_{2^m,2^n}(f)_{p,F} \le V_p(f)^*_{m,n}$..

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Keywords: functions of bounded p-fluctuation, Walsh systems, discrete modulus of continuity.

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Nijenhuis Geometry and Geodesically Equivalent Metrics

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Let M^n be a smooth manifold and L an operator filed on it.

Definition. Nijenhuis torsion N_L is a tensor type of $(1,2)$ which in the invariant form can be defined by following formula

$$N_L = L^2[u,v] + [Lu, Lv] - L[Lu, v] - L[u, Lv],$$

where u and v are a smooth vector fields.

Definition. Operator field L is Nijenhuis operator if $N_L \equiv 0$.

Definition. Two metrics g and $\bar{g}$ are called Geodesically Equivalent if they share same geodesics viewed as unparameterized curves.

Let g and $\bar{g}$ are metrics on the smooth manifold M. Then the condition of the Geodesic Equivalence can be written by the following equation:

$$\nabla^g_k R_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

where $R = |\frac{\det \bar{g}}{\det g}|^{\frac{1}{n+1}} g\bar{g}^{-1}g$, $\lambda_i = \frac{\partial R_{sj} g^{sj}}{\partial x^i}$.

In the coordinates it's a system of linear PDE for components of the metric g.

$$g_{is}\frac{\partial L^s_j}{\partial x^k} + g_{js}\frac{\partial L^s_i}{\partial x^k} - (\frac{\partial g_{sk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^s})L^s_j - (\frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s})L^s_i = \frac{\partial trL}{\partial x^i} g_{jk} + \frac{\partial trL}{\partial x^j} g_{ik}$$

Theorem 1.[1] $L = g^{-1}R$ is Nijenhuis operator.

Definition. A metric g and a Nijenhuis operator L are said to be *geodesically compatible*, if L is g-self-adjoint and the metric $\bar{g} = \frac{1}{\det L} gL^{-1}$ is geodesically equivalent to g.

Let $\chi(t) = \det(tE - L) = t^n + \sigma_1 t^{n-1} + \ldots + \sigma_n$ be characteristic polynomial of the Nijenhuis operator L.

Theorem 2. For geodesically compatible pair (L, g) true next formula:

$$\det g = f_1(\lambda_1)\ldots f_n(\lambda_n)(\det d\Phi)^2,$$

where λ_i are eigenvalues and $\Phi : M^n \to \mathbb{R}^n$, $\Phi = (\sigma_1, \ldots, \sigma_n)$, f_i are arbitrary almost everywhere smooth functions.

Theorem 3. Let L be a Nijenhuis operator, then:

1. If L is algebraically generic, then there exists geodesically compatible metric g and geodesic flow of the g is integrable;

2. If L is gl-regular ([1]), then also we can find geodesically compatible metric g and geodesic flow of the g is integrable;

In the talk I will show examples of the Nijenhuis operators when there does not exist geodesically compatible metric.

Keywords: Integrable systems, Nijenhuis geometry, Nijenhuis operators, Geodesically Equivalent Metrics, Geodesic flow.

2020 Mathematics Subject Classification: 53B99, 53A45, 37K10

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Radial and Logarithmic Refinements of the Weighted Hardy Inequality

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In this paper, we obtain weighted versions of radial and logarithmic refinements of the following Hardy's inequality using the factorization method of differential operators from [1]-[3]:

$$\int_{\Omega} |(\nabla f)(x)|^2 d^n x \geqslant \int_{\Omega} |x - x_0|^{-2} |f(x)|^2 \left\{ \frac{(n-2)^2}{4} + \frac{1}{4} \sum_{j=1}^{m} \prod_{k=1}^{j} \left[\ln_k \left(\gamma / |x - x_0|\right)\right]^{-2} \right\} d^n x, \tag{1}$$

valid for $f \in C_0^\infty(\Omega)$, assuming that $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}, n \geqslant 2$, is open and bounded with $x_0 \in \Omega, m \in \mathbb{N}$, and the logarithmic terms $\ln_k \left(\gamma / |x - x_0|\right), k \in \mathbb{N}$.

Moreover, we discuss generalizations of these results on homogeneous Lie groups.

This talk is based on the joint research with Nurgissa Yessirkegenov.

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Keywords: factorization method, Hardy's inequality, homogeneous Lie group, stratified group.

2010 Mathematics Subject Classification: 22E30,26D10

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Large Time Asymptotes to the Cauchy Problem for Doubly Nonlinear Parabolic Equation with Variable Density and Absorption

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The following Cauchy problem

$$\rho_2(x)\,\partial_t u = u^q div\left(\rho_1(x)\,u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\right) - \rho_3(x)\,t^l u^\beta, u(t_0,x) = u_0(x)\,,\ x \in R^N \quad (1)$$

is considered in $Q = \left\{(t,x) : \ t > 0,\ x \in R^N\right\}$,

where $m, k \geq 1, p \geq 2, n, l_i, l, \beta \ \ (i = 1,2)$ are the given numerical parameters

$$\rho_1(x) = |x|^n, \rho_{i+1}(x) = |x|^{-l_i},\ i = 1,2,\ \ q \in (0,1)\,.$$

The different qualitative properties of the solution problem (1) in one and many dimensional cases intensively studied by many authors [see [1-3] and references therein). In this work estimates of solutions and a free boundary, behaviors of solutions of the problem (1) for slowly $(k(p-2)+m-1>0)$, fast $(k(p-2)+m-1<0)$, a critical diffusion $(k(p-2)+m-1=2)$ cases, depending on value of density $p > n+l, p = n+l$ (singular) established. The problem choosing of an initial approximation for iteration process solved. The results of numerical experiments discussed.

Keywords: global solvability, estimate solution, critical, singular cases, asymptotic, numerical analysis..

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The Absolute Stable Difference Schemes to the Time-Dependent Source Identification Problem for the Schrödinger Equation

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Keywords: Schrödinger equation, source identification problem, Hilbert space, positive operator, absolute stable, difference schemes.

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On the Asymptotic Solutions of Perturbation Problems for Hyperbolic Equations

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The abstract initial value problem

$$\begin{cases} \varepsilon^2 u^{''}(t) + Au(t) = f(t), 0 < t < T, \\ u(0) = \varphi,\ u^{'}(0) = \psi \end{cases}$$

for hyperbolic equations is considered. The study is done in a Hilbert space H with the self adjoint positive definite operator A. A small parameter $\varepsilon \in (0, \infty)$ is used in the equation. The solution of the problem using an asymptotic formula with a small parameter is introduced. The study presents the general perturbation theory of uniform difference schemes on hyperbolic PDEs. Convergence estimes are obtained and some numerical verifications which verify the theoretical results are presented.

Keywords: hyperbolic equations, Cauchy problem, asymptotic formula, uniform difference schemes.

2020 Mathematics Subject Classification: 47A55, 39A30, 34K25

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Source Identification Problem for Multidimensional Reverse Parabolic Equation

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In the paper [1] source identification problem for parabolic equation with multipoint nonlocal condition was studied. The paper [2] is devoted to the well-posedness of direct nonlocal problems for a reverse parabolic equation with integral type nonlocal condition. The aim of the present work is to study the source identification problem for the multidimensional reverse parabolic equation.

We consider in $[0,1]\times\overline{\Omega}$, the source identification problem to find unkown functions v and p for the multi dimensional reverse parabolic equation with the following multi point nonlocal and Dirichlet boundary conditions

$$
(14) \qquad \begin{cases}
\frac{\partial v(t,x)}{\partial t}+\sum\limits_{i=1}^{n}(a_r(x)v_{x_i}(t,x))_{x_i}-\sigma v(t,x)=f(t,x)+p(x), \\
x=(x_1,\cdots,x_n)\in\Omega, \quad 0<t<1, \\
v(1,x)=\sum\limits_{k=1}^{r}\mu_k v(s_k,x)+\psi, \ v(0,x)=\varphi(x), x\in\overline{\Omega}, \\
v(t,x)=0, \quad x\in S.
\end{cases}
$$

Here $\Omega=(0,1)^n$ is the unit open cube in the $\mathbb{R}^n$ with boundary $S=\partial\Omega, \overline{\Omega}=\Omega\cup S$ and $\gamma_1,\gamma_2,...,\gamma_r,\mu_1,\mu_2,...,\mu_r$ are given numbers and inequalities

$$
(15) \qquad \sum_{k=1}^{r}|\mu_k|<1, 0\le\gamma_1<\gamma_2<...<\gamma_r<1.
$$

Let $a_r(x)\ge a_0>0, r=1,...,n$ be given functions, σ is a known positive number.

Theorem. Assume that the inequalities (15) are valid, $\varphi\in L_2(\Omega)$, $\psi\in W_2^2(\Omega)$, $f\in C^{\alpha}(L_2(\Omega))$ are given. Then, the source identification problem (14) is uniquely solvable and for its solution the following stability estimates are valid:

$$
(16) \qquad \|p\|_{L_2(\Omega)}\le M\left[\|\varphi\|_{L_2(\Omega)}+\|\psi\|_{W_2^2(\Omega)}+\frac{1}{\alpha}\|f\|_{C^{\alpha}(L_2(\Omega))}\right]
$$

$$
(17) \qquad \|v\|_{C(L_2(\Omega))}\le M\left[\|\varphi\|_{L_2(\Omega)}+\|\psi\|_{L_2(\Omega)}+\|f\|_{C(L_2(\Omega))}\right],
$$

where positive number M is independent of $f(t,x),\psi(x),\varphi(x)$, and α.

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Keywords: reverse parabolic equation, source identification problem, stability, inverse problem, multi point conditions, .

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Third-order weighted essentially nonoscillatory schemes for advection-diffusion equation

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Advection-diffusion equations are widely used in the modeling of many physical phenomena, such as in fluid dynamics, semiconductor physics, thermal heat diffusion, flow through porous media, the spread of pollution in rivers, etc. Since the analytical solutions of advection-diffusion equations are not easy to find, one needs to solve them by using numerical methods.

Weighted essentially non-oscillatory (WENO) schemes [1] have become popular techniques for the numerical solutions of hyperbolic conservation laws. It is known that the standard WENO schemes are unstable for advection-diffusion problems. This paper shows the extension of the third-order finite volume WENO scheme [2] to solve one-dimensional advection-diffusion equations.

Keywords: Advection-diffusion equation, WENO3 scheme

2020 Mathematics Subject Classification: 65M08, 65M20, 65D05

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Investigation of Discrete Analogs of Integral Geometry Problem with a Weight Function

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The interest in reconstruction problems has grown tremendously in the last four decades, stimulated by the spectrum of new modalities of image reconstruction. These are X-ray, MRI, gamma and positron radiography, ultrasound, seismic tomography, electron microscopy, synthetic radar imaging and others. The physical principles of these methods are very different; however their mathematical models and solution methods have very much in common. The umbrella name reconstructive integral geometry is used to specify the variety of these problems and methods [1]-[3].

With fairly general assumptions about the family of curves and the weight function, the problem of integral geometry is reduced to a boundary value problem for a second-order partial differential equation. Estimates of the conditional stability of discrete analogs of a two-dimensional integral geometry problem with a weight function on the space of sufficiently smooth functions are obtained.

The results of studying the problem under consideration contribute to the development of the theory and practical implementation of integral geometry problems in various fields of science and technology.

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Keywords: integral geometry, stability estimate, difference problem, finite-difference analogue, differential-difference analogue.

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On Some Nonlocal Boundary Value Problems for Second Order Elliptic Systems

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We consider an elliptical system

$$a_{11}\frac{\partial^2 u}{\partial x^2}+(a_{12}+a_{21})\frac{\partial^2 u}{\partial x\partial y}+a_{22}\frac{\partial^2 u}{\partial y^2}=0 \tag{1}$$

with constant coefficients $a_{ij}\in\mathbb{R}^{l\times l}$.

Its solution is understood as l - vector function $u=(u_1,\dots,u_l)$ from class C^2 in the area under consideration, satisfying identical to equation (1).
The ellipticity condition is that $\det a_{22}\neq 0$ and the characteristic polynomial

$$\chi(z)=\det P(z),\quad P(z)=a_{11}+(a_{12}+a_{21})z+a_{22}z^2, \tag{2}$$

of degree $2l$ have no real roots.
Let $\nu_1,\dots,\nu_m$ be all distinct roots of the polynomial χ, lying in the upper half-plane $\operatorname{Im} z>0$, which we also call the eigenvalues of the matrix polynomial $P(z)$.

Lemma 1. *There are matrices $b,J\in\mathbb{C}^{l\times l}$, such that*

$$\det\begin{pmatrix} b & \overline{b}\\ bJ & \overline{bJ}\end{pmatrix}\neq 0,\quad \sigma(J)=\{\nu_1,\dots,\nu_m\}, \tag{3}$$

$$a_{11}b+(a_{12}+a_{21})bJ+a_{22}bJ^2=0. \tag{4}$$

The 1st order canonical system is closely related to the elliptic system (1)

$$\frac{\partial\phi}{\partial y}-J\frac{\partial\phi}{\partial x}=0,\ J=\operatorname{diag}\,(J_1,\dots,J_m)\in\mathbb{C}^{l\times l},\ J_i\in\mathbb{C}^{l_i\times l_i},\ \sigma(J_i)=\{\nu_i\} \tag{5}$$

which, in particular, for $J=i$ corresponds to the Cauchy-Riemann system.

For equation (1) in the half-plane $D=\{\,y>0\}$, consider the boundary value problem

$$(pu_x+qu_y+p_0u)\big|_{\mathbb{R}}=g. \tag{6}$$

We search the solution u in the class $C^{1,\mu}_{(\lambda)}(D;F)$, $0<\lambda<1$.
The right side g is selected from the class $C^{\mu}_{\lambda-\sigma}(D;F)$, $l\times l$-the matrix coefficients p,q and p_0 are assumed to be piecewise continuous on $\mathbb{R}$ with possible discontinuities at points $\tau\in F$. More precisely, there exists a $\varepsilon>0$, such that

$$p,q\in C^{\mu+\varepsilon}_{(\varepsilon)}(G;a,b),\quad p_0\in C^{\mu+\varepsilon}_{\varepsilon-\sigma}(G;a,b) \tag{7}$$

for any interval $G=(a,b)$, into which the complement $\mathbb{R}\setminus F_0$. is divided.

Theorem 1. *Problem (1), (6) in class $\{u\in C^{1,\mu}_{(\lambda)}, u(\infty)=0\}$, $0<\lambda<1$, is equivalent to finding analytic vector functions in this class $\psi(z)$ by boundary condition*

$$\operatorname{Re}\,(G\psi'+G_0\psi)\big|_{\mathbb{R}}=g \tag{8}$$

with matrix coefficients $G=pb+qbJ$, $G_0=p_0b$.
The connection between the solutions of these problems is carried out by the Bitsadze formula [1]

$$u=\operatorname{Re}\,bE\psi, \tag{9}$$

where the operation E is given block by block by the equality

$$(E\psi)_i=E_i\psi_i=\sum_{k=0}^{l_i-1}\frac{y^k}{k!}(J_i-\nu_i)^k\psi_i^{(k)}(x+\nu_iy),\ z=x+iy,\ i=1,\dots,m. \tag{10}$$

Let us consider the case when the coefficients p, q in (6) are constant, and $p_0 = 0$, i.e. the problem

$$(pu_x + qu_y)\big|_{\mathbb{R}} = g, \quad p, q \in \mathbb{R}^{l\times l}. \tag{11}$$

In this case, the matrix $G = pb + qbJ \in \mathbb{C}^{l\times l}$ and the statements of Theorem 1 can be significantly supplemented.

Theorem 2. *a) If* $\det G = 0$, *then homogeneous problem (1), (11) in the class* $C^{1,\mu}_{(\lambda)}$, $0 < \lambda < 1$, *has an infinite number of linearly independent solutions. For example, if* $G\eta = 0$ *for some nonzero* $\eta \in \mathbb{C}^l$, *then the solutions to the homogeneous problem are the functions* $u = \text{Re } b(E\psi_0)\eta$ *for any scalar analytic function* $\psi_0 \in C^{1,\mu}_{(\lambda)}$.
b) If $\det G \neq 0$, *then the condition*

$$\int_{\mathbb{R}} g(t)dt = 0 \tag{12}$$

is necessary and sufficient for the solvability of an inhomogeneous problem in the class $C^{1,\mu}_{(\lambda)}$, $0 < \lambda < 1$. *Under the additional assumption* $u(\infty) = 0$ *its solution is unique and is given by the formula*

$$u(z) = \text{Re } \frac{1}{\pi}\int_{\mathbb{R}} b[t-z]^{-1}G^{-1}f(t)dt, \quad f(x) = \int_{-\infty}^{x} g(t)dt. \tag{13}$$

We introduce the concept of the conjugate function v to the solution u of equation (1). It is determined by the line integral

$$v(z) = v(z_0) + \int_{z_0}^{z} (a_{21}u_x + a_{22}u_y)dx - (a_{11}u_x + a_{12}u_y)dy.$$

For v we get the representation

$$v = \text{Re } c\phi, \quad c = a_{21}b + a_{22}bJ. \tag{14}$$

Together with $u(z)$ this function also belongs to the class $C^{1,\mu}_{(\lambda)}$, $0 < \lambda < 1$. If we put

$$\begin{pmatrix} b & \bar{b} \\ c & \bar{c} \end{pmatrix}^{-1} = \begin{pmatrix} b_2 & c_2 \\ \bar{b}_2 & \bar{c}_2 \end{pmatrix}$$

for its inverse matrix, we can write the formula

$$\phi = 2(b_2u + c_2v). \tag{15}$$

From (11) we can go to the "integrated" boundary value problem

$$(pu + qv)\big|_{\mathbb{R}} = f \tag{16}$$

in class $C^{1,\mu}_{(\lambda)}$, $0 < \lambda < 1$, where p_1, q_1 are again denoted by p, q.
As a simple example, consider for system (1) the boundary value problem

$$u(x,0) + u(-x,0) = f(x), \quad v(x,0) + v(-x,0) = g(x), \tag{17}$$

where f, g are given even l -vector functions.

Theorem 3. *Problem (1), (17) in class* $u + iv \in C^{\mu}_{(\lambda)}(D,\ F)$, $0 < \lambda < 1$, *is uniquely solvable and its solution is given by the formula*

$$u(z) = \text{Re } \frac{1}{\pi i}\int_{\mathbb{R}} b[t-z]^{-1}h(t)dt, \quad h = b_2f + c_2g, \tag{18}$$

where matrices b_2, c_2 *appear in (15).*
In [2], the Fredholm solvability of boundary value problems for elliptic systems was studied. In [3], the Fredholm solvability of boundary value problems for high-order elliptic equations was studied and the index formula for the problem was calculated.

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Keywords: elliptic systems, boundary value problem, Fredholm solvability of the problem, problem index formula.

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Weak Version of Symmetric Space

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Let E_i be a (quasi) symmetric Banach space on $(0,1)$, $i = 1,2$. We define the pointwise product space $E_1 \odot E_2$ as

$$E_1 \odot E_2 = \{f : \ f = f_1 f_2, \ f_i \in E_i, \ i = 1, \ 2\} \tag{18}$$

with a functional $\|f\|_{E_1 \odot E_2}$ defined by

$$\|f\|_{E_1 \odot E_2} = \inf\{\|f_1\|_{E_1}\|f_2\|_{E_2} : \ f = f_1 f_2, \ f_i \in E_i, \ i = 1, \ 2\}.$$

Definition 1. *Let E be a symmetric (quasi) Banach space on $(0,1)$. The fundamental function φ_E is defined by $\varphi_E(t) = \|\chi_A\|$, where $t \in [0,1)$ and A is a measurable subset of $(0,1)$ with $m(A) = t$.*

Let $M_{\varphi_E}(0,1)$ be the usual Marcinkiewicz spaces with norms defined by

$$M_{\varphi_E}(0,1) = \{f \in L_0(0,1) : \ \|f\|_{M_{\varphi_E}} = \sup_{t>0} \frac{\varphi_E(t)}{t} \int_0^t \mu_s(f) ds < \infty\}.$$

Definition 2. *Let E be a symmetric (quasi) Banach space on $(0,1)$. We call $M_{\varphi_E}(0,1)$ is weak version of E and denote it by E_∞.*

The classical weak L_p-space $L_{p,infty}(0,1)$ $(1 \le p < \infty)$ is defined as the set of all measurable functions f on $(0,1)$ such that

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} \mu_t(f) < \infty.$$

For $p > 1$, $L_{p,\infty}(0,1)$ can be renormed as a Banach space by

$$f \mapsto \sup_{t>0} t^{-1+\frac{1}{p}} \int_0^t \mu_s(f) ds.$$

If $E = L_p(0,1)$ $(1 < p < \infty)$, then $E_\infty = L_{p,\infty}(0,1)$. If $(L_p(0,1))_\infty = L_p(0,1)$. But for $0 < p \le 1$, if $f \in (L_p(0,1))_\infty$, then

$$\|f\|_{(L_p(0,1))_\infty} = \sup_{t>0} t^{\frac{1}{p}-1} \int_0^t \mu_s(f) ds = \int_0^1 \mu_s(f) ds = \|f\|_1.$$

Hence, $(L_p(0,1))_\infty = L_1(0,1)$ and it is different from the classical weak L_p-space.

Let Φ be an N-function, we define

$$a_\Phi = \inf_{t>0} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi = \sup_{t>0} \frac{t\Phi'(t)}{\Phi(t)}.$$

If $b_\Phi < \infty$, then the fundamental function of Orlicz space $L_\Phi(0,1)$ on $(0,1)$ equipped with the Luxemburg norm, is the following

$$\varphi_{L_\Phi(\Omega)}(t) = 1/\Phi^{-1}(\frac{1}{t}), \quad t > 0.$$

Hence, if $E = L_\Phi(0,1)$ and $1 < a_\Phi \le b_\Phi < \infty$, then $E_\infty = L_{\Phi,\infty}(0,1)$.

Proposition 1. *Let E_i be symmetric (quasi) Banach space on $(0,1)$ which is α_i-convex for some $0 < \alpha_i < \infty$ $(i = 1, \ 2)$. Then E_1 and E_2 can be equipped with equivalent quasi norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively, so that $\varphi_{E_1 \odot E_2}(t) = \varphi_{E_1}(t)\varphi_{E_2}(t)$, for any $t \ge 0$.*

Theorem 1. *Let E_i be symmetric (quasi) Banach space on $(0,1)$ which is α_i-convex for some $0 < \alpha_i < \infty$ $(i = 1, \ 2)$ and $0 < a < 1$. If $x \in ((E_1^{(a)})_\infty)^{(\frac{1}{a})}(\mathcal{M})$ and $y \in ((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}(\mathcal{M})$, then $xy \in (E_1 \odot E_2)_\infty(\mathcal{M})$ and the following Holder type inequality holds*

$$\|xy\|_{(E_1 \odot E_2)_\infty} \le \|x\|_{((E_1^{(a)})_\infty)^{(\frac{1}{a})}} \|y\|_{((E_2^{(1-a)})_\infty)^{(\frac{1}{1-a})}}.$$

Theorem 2. *Let E be symmetric (quasi) Banach space on $(0,1)$. Then we have the following Chebyshev type inequality*

$$t\varphi_E(\tau(e_{(t,\infty)}(|x|))) \leq \|x\|_{E\infty}, \quad \forall x \in E_\infty(\mathcal{M}).$$

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Keywords: symmetric space, fundamental function of symmetric space, noncommutative symmetric space, von Neumann algebra.

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Initial-Boundary Value Problems for a Two-Dimensional Wave Equation with Nonlocal Conditions which are Multidimensional Generalizations of the Samarskii-Ionkin Problem

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It is well known that initial boundary value problems for the wave equation often arise in the context of physical systems that describe the propagation of waves, such as sound waves, water waves, electromagnetic waves, and others. These problems typically involve a combination of initial conditions and boundary conditions for a differential equation.

Problems with nonlocal boundary conditions represent a special class of initial boundary value problems in which the boundary conditions simultaneously connect the values of the desired solution (and its derivatives) at different points of the boundary. Such conditions can occur in a variety of physical and mathematical contexts and can describe more complex physical phenomena and boundary conditions.

The report discusses the formulation of new initial-boundary value problems for a two-dimensional wave equation with nonlocal conditions respect to spatial variables, which are multidimensional generalizations of the Samarsky-Ionkin problem. The domain of consideration of the problem is a circular cylinder Q with its axis along the t axis. Classical initial conditions are set at the base of the cylinder and new nonlocal boundary conditions on the spatial (lateral) boundaries of the cylinder.

Let $\Omega = \{(r,\varphi) : 0 \le r < 1, 0 \le \varphi \le 2\pi\}$ be the unit circle, $Q = \{(r,\varphi,t) : 0 \le r < 1, 0 \le \varphi \le 2\pi, 0 < t < T\}$ is a right circular cylinder.

We will consider a new nonlocal boundary value problem for the two-dimensional wave equation:

$$u_{tt}(r,\varphi,t) - \Delta u(r,\varphi,t) = f(r,\varphi,t),\ (r,\varphi,t) \in Q, \tag{1}$$

where Δ is the Laplace operator in polar coordinates (r,φ).

We will use classical initial conditions

$$u|_{t=0} = \tau(r,\varphi),\ u_t|_{t=0} = \nu(r,\varphi),\ (r,\varphi) \in \Omega, \tag{2}$$

and nonlocal boundary conditions on the lateral boundary of a circular cylinder

$$u(1,\varphi,t) - \alpha u(1,2\pi-\varphi,t) = 0,\ 0 \le \varphi \le \pi,\ 0 \le t \le T, \tag{3}$$

$$\frac{\partial u}{\partial r}(1,\varphi,t) - \frac{\partial u}{\partial r}(1,2\pi-\varphi,t) = 0,\ 0 \le \varphi \le \pi, \le t \le T. \tag{4}$$

Here $\alpha \ne 1$ is a fixed real number.

We take the right side of the equation and the initial conditions from the following "standard" smoothness class for hyperbolic problems: $f(r,\varphi,t) \in C^{1+\epsilon}(\bar{Q})$; $\tau(r,\varphi) \in C^{2+\epsilon}(\bar{\Omega})$; $\nu(r,\varphi) \in C^{1+\epsilon}(\bar{\Omega})$. Additionally, we require that $\tau(r,\varphi)$ and $\nu(r,\varphi)$ satisfy boundary conditions (3), (4).

To solve the initial-boundary-value problem (1)-(4), we apply the method of reducing to a sequential solution of two initial-boundary value problems with self-adjoint boundary conditions in the spatial variable, proposed in [1] for the case of one-dimensional parabolic initial-boundary value problems with non-reinforced regular boundary value problems conditions.

The main result of the work is the proof of the correctness of the formulated problem in the classical sense.

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Keywords: multidimensional wave equation, nonlocal boundary condition, Samarskii-Ionkin condition, classical solution.

2020 Mathematics Subject Classification: 35L05, 35L20

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On the Compactness of the Commutator of the Riesz Potential in Global Morrey Type Spaces

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In this work a sufficient conditions for the boundedness and compactness of the commutator of the Riesz Potential in global Morrey-type spaces are obtained.

DEFINITION. Let $0 < p, \theta \leq \infty$ and let w be a nonnegative measurable function on $(0, \infty)$. The general global Morrey-type space $GM_{p\theta,w(\cdot)} \equiv GM_{p\theta,w(\cdot)}(\mathbb{R}^n)$ is defined as the set of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ with finite quasi-norm

$$\|f\|_{GM_{p\theta,w(\cdot)}} \equiv \sup_{x\in\mathbb{R}^n} \left\| w(r) \, \|f\|_{L_p(B(x,r))} \right\|_{L_\theta(0,\infty)},$$

where $B(t,r)$ the ball with center at the point t and of radius r.

We denote by $\Omega_{p\theta}$ the set of all functions that are nonnegative, measurable on $(0,\infty)$, not equivalent to 0 and such, that for some $t > 0$ (and therefore for all $t > 0$)

$$\|w(r) r^{\frac{n}{p}}\|_{L_\theta(0,t)} < \infty, \quad \|w(r)\|_{L_\theta(t,\infty)} < \infty.$$

The space $GM_{p\theta,w(\cdot)}$ is non-trivial, that is, it consists not only of functions equivalent to 0 in $\mathbb{R}^n$ if and only if $w \in \Omega_{p\theta}$.

We consider the Riesz Potential $I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$

For a function $b \in L_{loc}(\mathbb{R}^n)$ by M_b denote multiplier operator $M_b f = bf$, where f-measurable function. Then the commutator between I_α and M_b is defined by

$$[b, I_\alpha] = M_b I_\alpha - I_\alpha M_b = \int\limits_{\mathbb{R}^n} \frac{(b(x) - b(y)) \, f(y)}{|x-y|^{n-\alpha}} dy.$$

Let u, v be weight functions. Denote by

$$H^* g(t) := \int_t^\infty g(s) ds, \quad g \in \mathfrak{M}^+,$$

the Hardy operator,

$$W(t) := \int_0^t w(t) dw,$$
$$U_*(t) := \int_t^\infty u(t) du,$$
$$V_*(t) := \int_t^\infty v(t) dv.$$

Theorem 1. Let $1 < p \leq q < \infty$, $0 < \alpha < n$ and $b \in BMO(R^n)$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w_1, w_2 \in \Omega_\theta$ and let the functions w_1, w_2 satisfy the conditions

$$A_0^* := sup_{t>0} \left(\int_t^\infty \int_\tau^\infty (1 + ln\frac{\tau}{r}) dr w_1(\tau) d\tau \right)^{\frac{1}{q}} \left(\int_t^\infty v(t) dv \right)^{-\frac{1}{p}} < \infty,$$

$$A_1^* := sup_{t>0} W_2^{\frac{1}{q}}(t) \left(\int_t^\infty (\frac{U_*(\tau)}{V_*(\tau)})^{p'} v(\tau) d\tau \right)^{\frac{1}{p'}} < \infty.$$

Then the commutator $[b, I_\alpha]$ is the boundedness operator from $GM_{p\theta}^{w_1}$ to $GM_{q\theta}^{w_2}$.

Theorem 2. Let $1 < p \leq q < \infty$, $0 < \alpha < n$ and $b \in VMO(\mathbb{R}^n)$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $w_1, w_2 \in \Omega_\theta$ satisfy the conditions of Theorem 1.

Then the commutator $[b, I_\alpha]$ is a compact operator from $GM_{p\theta}^{w_1}$ to $GM_{q\theta}^{w_2}$.

Remark. The compactness of the commutator of the Riesz potential in Morrey space was considered in [1]. The compactness of sets in global Morrey-type spaces was studied in [2], [3].

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Keywords: Compactness, Commutators, Riesz Potential, Global Morrey-type spaces.

2020 Mathematics Subject Classification: 35Q79, 35K05, 35K20.

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Inverse Source Problem for Time-Fractional Diffusion Equation with Nonlocal Boundary Condition

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In this paper, we consider the inverse problem of finding a pair of functions $\{p(t), u(x,t)\}$ such that it satisfies the equation

$$ {}_0\partial_t^\alpha u = u_{xx} + p(t)f(x,t),\ (x,t) \in D_T, \tag{1}$$

the initial condition

$$ u(x,0) = \varphi(x),\ 0 \le x \le 1, \tag{2}$$

the boundary conditions

$$ u(0,t) = 0,\ u_x(0,t) + du_{xx}(1,t) = 0,\ 0 \le t \le T, \tag{3}$$

and the overdetermination condition

$$ \int_0^1 u(x,t)dx = E(t),\ t \in [0,T], \tag{4}$$

where $T > 0$, $D_T = \{(x,t) :\ 0 < x < 1,\ 0 < t \le T\}$, f, φ are given real-valued functions, $d > 0$ is a real constant, ${}_0\partial_t^\alpha u$ is the Caputo time-fractional derivative of order α defined in [1] by

$$ {}_0\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha},\ 0 < \alpha < 1,$$

and $\Gamma(\cdot)$ is the Gamma function.

In the case when $\alpha = 1$, the inverse problem of finding pairs $\{p(t), u(x,t)\}$ with the conditions (2)-(4) for the equation

$$ u_t = u_{xx} + p(t)u + f(x,t),\ (x,t) \in D_T$$

is considered in [2].

The well-posedness of the inverse problem (1)-(4) is shown by Fourier expansion in terms of eigenfunctions of the spectral problem, which has a spectral parameter in the boundary condition. Also, for the well-posedness of the problem (1)-(4), the properties of the Volterra integral equation of the second kind have been used. One of the main results of this work is that the existence and uniqueness of the solution of the problem (1)-(4) has been proved without using the orthogonality condition on the input data. The continuous dependence of the solution of the problem (1)-(4) on the data has been proved.

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Keywords: inverse source problem, fractional diffusion equation, nonlocal boundary condition, problem with spectral parameter in boundary condition, Mittag-Lefler type function.

2020 Mathematics Subject Classification: 35K20, 35R30, 35A09

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On a Nonlocal Frankl-Type Boundary Value Problem for an Equation of Mixed Parabolic-Hyperbolic Type

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In this talk a new nonlocal boundary value problem for an equation of the mixed type is formulated. This equation is parabolic-hyperbolic and belongs to the first kind because the line of type change is not a characteristic of the equation. Nonlocal condition binds points on boundaries of the parabolic and hyperbolic parts of the domain with each other. This problem is generalization of the well-known problems of Frankl type. A boundary value problem for the heat equation with conditions of the Samarskii-Ionlin type arises in solving this problem. Unlike the existing publications of the other authors related to the theme it is necessary to note that in this papers the nonlocal problems were considered in rectangular domains. But in our formulation of the problem the hyperbolic part of the domain coincides with a characteristic triangle. Unique strong solvability of the formulated problem is proved.

Let $\Omega \subset R^2$ be a finite domain bounded for $y > 0$ by the segments AA_0, A_0B_0, B_0B, $A = (0, 1)$, $B_0 = (1, 1)$, $B = (1, 0)$ and for $y < 0$ by the characteristics $AC: \ x + y = 0$ and $BC: \ x - y = 1$ of an equation of the mixed parabolic-hyperbolic type

$$Lu = \left\{ \begin{array}{c} u_x - u_{yy}, y > 0 \\ \\ u_{xx} - u_{yy}, y < 0 \end{array} \right\} = f(x, y). \tag{1}$$

This is the equation of the mixed type. The equation refers to the first kind because the line of change of type $y = 0$ is not a characteristic of the equation.

By $W_2^l(\Omega) = H^l(\Omega)$ we denote the space of S.L. Sobolev with the scalar product $(\cdot, \ \cdot)_l$ and the norm $\|\cdot\|_l$, $W_2^0(\Omega) = L_2(\Omega)$; $\Omega_1 = \Omega \bigcap \{y > 0\}$, $\Omega_2 = \Omega \bigcap \{y < 0\}$.

In Ω consider the following nonlocal boundary value problem being the generalization of an analogue of the Frankl problem for the parabolic-hyperbolic equation (1).

Problem F. *Find a solution to Eq. (1) satisfying classical boundary conditions*

$$u|_{AA_0} = 0, \quad u_y|_{A_0B_0} = 0 \tag{2}$$

and a nonlocal boundary condition

$$u(\theta(t)) = au(\theta_0(t)) + bu(\theta_1(t)), \quad 0 \le t \le 1, \tag{3}$$

where $\theta(t) = (t, 1)$, $\theta_0(t) = \left(\frac{t}{2}, -\frac{t}{2}\right)$, $\theta_1(t) = \left(\frac{t+1}{2}, \frac{t-1}{2}\right)$; a, b *are arbitrary numbers.*

It is easy to see that $\theta(t) \in A_0B_0$, $\theta_0(t) \in AC$, $\theta_1(t) \in BC$. Therefore the new nonlocal boundary condition (3) binds with each other values of the sought-for solution on the parabolic part of the boundary A_0B_0 and on the hyperbolic parts of the boundary of the domain (at the characteristics AC and BC).

Theorem. *Let* $a + b \neq 0$. *Then for any function* $f \in L_2(\Omega)$ *there exists a unique strong solution* $u(x, y)$ *to the problem* F. *This strong solution belongs to the class* $H^1(\Omega) \cap H_{x,y}^{1,2}(\Omega_1) \cap C(\overline{\Omega})$, *and satisfies the inequality*

$$\|u\|_1 \le C \|f\|_0.$$

Note that in the special case when $a = 1 + \alpha$ and $b = 1 - \alpha$ (α is a real number), the problem F was considered in our paper [1].

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Keywords: equation of the mixed type, nonlocal boundary condition, Frankl problem, strong solution, conditions of the Samarskii-Ionlin type.

2020 Mathematics Subject Classification: 35M12

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Solvable Problems for the Laplace-Beltrami Operator on a Two-Dimensional Sphere with a Cut

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It is known [1] that the two-dimensional sphere S in the three-dimensional space $\mathbb{R}^3$ represents a Riemannian manifold. The Laplace-Beltrami operator Δ_S on the indicated sphere in a standard way is introduced. Since the sphere S is a smooth manifold without boundary, the problem is well defined.

$$u(x) - \Delta_S u(x) = f(x), \quad x \in S.$$

The sphere S of the closed curve C is split into two non-intersecting parts S_2 and S_1 [2],[3].

The following Dirichlet problem studied in the work [2]

$$-\Delta_S u(x) = 0, \quad x \in S_2, \quad u(x) = g(x), \quad x \in C.$$

Using the potentials of a simple and double layer, the existence of a solution to the indicated Dirichlet problem is proved.

Another problem is studied in the work [3]

$$-\Delta_S u(x) = \omega(x), \quad x \in S_2, \tag{1}$$

$$-\frac{1}{2}u(x) + \int_C u(y)\underline{curl}_S \varepsilon(x,y) \cdot \vec{t}(y) ds_y -$$
$$- \int_C \varepsilon(x,y)\underline{curl}_S u(y) \cdot \vec{t}(y) ds_y = 0, \quad x \in C. \tag{2}$$

Here $\varepsilon(x,y) = -\frac{1}{4\pi}\ln|1 - \langle x, y\rangle|$ represents the fundamental solution of the Laplace-Beltrami operator. It is shown in the work [3], that the solution to problem (1),(2) is written as

$$u(x) = \int_{S_2} \varepsilon(x,y)\omega(y) d\sigma_y, \quad x \in S_2.$$

We also note the paper [4], where a boundary value problem for the Laplace-Beltrami operator on a punctured sphere was studied. A punctured sphere is a sphere from which one point has been removed. In this case, the problem arises: What additional conditions must the solution at the remote point satisfy in order to guarantee the uniqueness of the solution?

In the present paper, a fixed arc from the sphere is removed and the same question is studied: what additional conditions must the solution on the removed arc satisfy in order to guarantee the uniqueness of such a solution?

To solve this problem we introduce a class of the function $W^2_{2,loc}(S_{20}) = \bigcup\limits_{\delta>0} W^2_2(S_{2\delta})$.

As well as necessary to introduce a class of the function

$$W^2_{2L}(S_{20}) = \{h \in W^2_{2,loc}(S_{20}) : \lim_{\delta\to 0}\int_{S_{20}} h(y)d\sigma_y = 0,$$
$$(L_d h)(x) - (L_s h)(x) \in H(S_{20}), \Delta_S h \in L_2(S), \}.$$

Here $L_s h$ and $L_d h$ are single and double layer potentials respectively, $h(x)$ is an arbitrary function from the class $W^2_{2L}(S_{20})$. Since $\Delta_S h \in L_2(S)$, that $T(x) \in W^2_2(S)$

and $\Delta_S T = \Delta_S h$. Thus, $T(x)$ represents a regularization of the function $h(x)$. In fact, the function $h(x)$ could have singularities on the curve C. At the same time, its regularization $T(x)$ no longer has singularities on the full sphere S, that is, it belongs to the space $W_2^2(S)$.

Lemma 1. *An arbitrary element h of the class $W_{2L}^2(S_{20})$ for $x \in S_{20}$ can be represented as $h(x) = T(x) + (L_s h)(x) - (L_d h)(x)$, where $T(x)$ is a function from $W_2^2(S)$.*

Now we can formulate the statement.

Theorem 1. *For any function $f \in L_2(S)$ and any function g from $H(S_{20})$, the problem*

$$-\Delta_S u(x) = f(x), \quad x \in S_{20},$$
$$(L_d u)(x) - (L_s u)(x) = g(x), \quad x \in S_{20}$$

has a unique solution in the class $W_{2L}^2(S_{20})$.

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Keywords: Laplace-Beltrami operator, single and double layer potentials, two-dimensional sphere.

2020 Mathematics Subject Classification: 31A15, 47B40, 31C12

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Global Existence and Blow-up of Solutions to Porous Medium Equation and Pseudo-Parabolic Equation Related to Baouendi-Grushin Operator

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In this talk, we discuss a global existence and blow-up of the positive solutions to the initial-boundary value problem of the nonlinear porous medium equation and the nonlinear pseudo-parabolic equation related to the Baouendi-Grushin operator. Our approach is based on the Poincaré inequality from [1] for the Baouendi-Grushin vector fields and the concavity argument.

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Keywords: Blow-up, porous medium equation, global solution, pseudo-parabolic equation, Baouendi-Grushin operator.

2010 Mathematics Subject Classification: 35K91, 35B44, 35A01

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On Pantograph Delay Differential Equations

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In this talk, we prove by means of some fixed point theorems, the existence and uniqueness of solutions for a class of fractional pantograph delay differential equations with initial conditions. Then, we investigate the stability and Ulam-Hyers stability for this type of delay differential equation. Furthermore, we apply the Adomian decomposition method to analyze numerically some examples.

Keywords: Fixed point Theorem, Fractional derivative, initial condition, Pantograph equation, Adomian decomposition method.

2020 Mathematics Subject Classification: 26A33, 34A08, 34B15.

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Construction of an Unconditional Basis from Elements of the System of Root Functions of one Spectral Problem with a Linear Occurrence of the Spectral Parameter in the Boundary Value Condition

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It is well known that system of eigenfunctions of an operator defined formally by a self-adjoint differential expression with arbitrary self-adjoint boundary value conditions, providing a discrete spectrum, forms an orthogonal basis of the space L_2. In the case when the spectral parameter is also present in the boundary value condition, consideration of the problem in the form of an operator in L_2 becomes impossible, since the spectral parameter should not change the domain of the operator. Problems of this type naturally arise when we solve (by the method of separation of variables) initial-boundary value problems for evolutionary equations, when the boundary value condition includes derivatives of the solution with respect to the time variable. For such problems, based on the theory developed by A.A. Shkalikov [1], it is known that when we remove (some finite number of) elements, the system of root vectors of such a problem can already form a basis with brackets. However, the question of whether the system forms an unconditional basis must be investigated for each problem separately. We consider a spectral problem with a linear occurrence of the spectral parameter in one of the boundary conditions

$$l(u) \equiv -u''(x) = \lambda u(x),\ 0 < x < 1, \tag{1}$$

$$u'(0) = 0,\ u'(1) = \lambda(u(0) + u(1)). \tag{2}$$

Representing the general solution of the equation (1) by the formula:

$$u(x, \lambda) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$$

and satisfying it by the boundary value conditions (2), we obtain a linear system with respect to the coefficients C_1 and C_2:

$$\begin{cases} C_2 = 0, \\ C_1\left(\sqrt{\lambda}\left[\sqrt{\lambda}(1+\cos\sqrt{\lambda}) + \sin\sqrt{\lambda}\right]\right) = 0. \end{cases} \tag{3}$$

Therefore, the characteristic determinant of the problem (1), (2) has the form

$$\Delta(\lambda) = \sqrt{\lambda}(1+\cos\sqrt{\lambda}) + \sin\sqrt{\lambda} = \cos\frac{\sqrt{\lambda}}{2}\left(\sqrt{\lambda}\cos\frac{\sqrt{\lambda}}{2} + \sin\frac{\sqrt{\lambda}}{2}\right).$$

Solving the equation $\Delta(\lambda) = 0$, we have two series of eigenvalues

$$\cos\frac{\sqrt{\lambda}}{2} = 0 \ \Rightarrow\ \sqrt{\lambda_k} = (2k+1)\pi \Rightarrow \lambda_k^{(1)} = ((2k+1)\pi)^2;$$

$$\sqrt{\lambda}\cos\frac{\sqrt{\lambda}}{2} + \sin\frac{\sqrt{\lambda}}{2} = 0 \ \Leftrightarrow\ \cot\frac{\sqrt{\lambda}}{2} = -\frac{1}{\sqrt{\lambda}}.$$

Due to the Rouche theorem, we get

$$\frac{\sqrt{\lambda}}{2} = \left(\frac{\pi}{2} + k\pi\right) + \frac{\delta_k}{2} \ \Rightarrow\ \sqrt{\lambda_k} = (2k+1)\pi + \delta_k,$$

δ_k are bounded. Let us prove that $\delta_k \to 0$.

$$\cot\left(\frac{\pi}{2} + k\pi + \frac{\delta_k}{2}\right) = -\frac{1}{(2k+1)\pi + \delta_k},$$

$$\cot\left[\left(\frac{\pi}{2}+k\pi\right)+\frac{\delta_k}{2}\right]=\frac{\cot\left(\frac{\pi}{2}+k\pi\right)\cdot\cot\frac{\delta_k}{2}-1}{\cot\left(\frac{\pi}{2}+k\pi\right)+\cot\frac{\delta_k}{2}}=\frac{-1}{\cot\frac{\delta_k}{2}}=\frac{1}{(2k+1)\pi+\delta_k}\ \Leftrightarrow$$

$$\Leftrightarrow\ \tan\frac{\delta_k}{2}=\frac{1}{(2k+1)\pi+\delta_k}=O\left(\frac{1}{k}\right)\ \Rightarrow$$

$\Rightarrow \delta_k=O\left(\frac{1}{k}\right)$. Thus, $\sqrt{\lambda_k}=(2k+1)\pi+O\left(\frac{1}{k}\right)$, that is $\lambda_k^{(1)}=((2k+1)\pi+\delta_k)^2$.

$\lambda_k^{(1)}$, $\lambda_k^{(2)}$ are simple (single) eigenvalues of the problem (1)-(2). From the system of equations (3): $C_2=0$, therefore, from the general solution of the equation (1), the system of eigenfunctions of the problem (1)-(2) is explicitly calculated and has the form:

$$u_k^{(1)}(x)=\cos((2k+1)\pi)(x),$$

$$u_k^{(2)}(x)=\cos((2k+1)\pi+\delta_k)(x),\ k=0,1,2,3,\dots,\text{ where, }\delta_k=O\left(\frac{1}{k}\right). \tag{4}$$

Due to the asymptotics of δ_k, by removing one element from this system, we obtain the complete and minimal system in $L_2(0,1)$. For large enough $k>>1$, this system does not form an unconditional basis, since two series of eigenfunctions $u_k^{(1)}(x)$ and $u_k^{(2)}(x)$ "stick together", although according to the results of A.A. Shkalikov [1], this system forms a Riesz basis with brackets in $L_2(0,1)$.

We introduce the following auxiliary system of functions:

$$y_k^{(1)}(x)=u_k^{(1)}(x),$$

$$y_k^{(2)}(x)=\frac{1}{\delta_k}(u_k^{(2)}(x)-u_k^{(1)}(x)),\ \ k=0,1,2,3,\dots, \tag{5}$$

that is, the auxiliary system of functions has the form

$$y_k^{(1)}(x)=\cos(2k+1)\pi x,$$

$$y_k^{(2)}(x)=-\frac{\sin(\delta_k x)}{\delta_k}\sin(2k+1)\pi x+\frac{\cos(\delta_k x)-1}{\delta_k}\cos(2k+1)\pi x,\ \ k=0,1,2,3,\dots. \tag{6}$$

We consider the system of functions

$$\{\varphi_k^{(1)}(x)=\cos(2k+1)\pi x,\ \varphi_k^{(2)}(x)=-x\sin(2k+1)\pi x\}_{k=0}^{\infty}. \tag{7}$$

These functions are eigenfunctions ($\varphi_k^{(1)}(x)$) and adjoint functions ($\varphi_k^{(2)}(x)$) of the Samarsky-Ionkin type problem:

$$-\varphi''(x)=\lambda\varphi(x),\ 0<x<1,$$

$$\varphi(0)+\varphi(1)=0,\ \varphi'(1)=0.$$

As shown in [2], the system of eigenfunctions and associated functions of this problem forms the Riesz basis in $L_2(0,1)$.

Let us show that the system of functions $\{y_k^{(1)}(x),y_k^{(2)}(x)\}$ and $\{\varphi_k^{(1)}(x),\varphi_k^{(2)}(x)\}$ are quadratically close.

Indeed, we get:

$$\sum_{k=1}^{\infty}\|\varphi_k-y_k\|^2=\sum_{k=1}^{\infty}\|\varphi_k^{(2)}-y_k^{(2)}\|^2.$$

By direct calculation, we obtain the estimation:

$$\|\varphi_k^{(2)}-y_k^{(2)}\|=$$

$$=\left\|-x\sin(2k+1)\pi x+\frac{\sin(\delta_k x)}{\delta_k}\sin(2k+1)\pi x-\frac{\cos(\delta_k x)-1}{\delta_k}\cos(2k+1)\pi x\right\|=$$

$$=\left\|\sin(2k+1)\pi x\left(\frac{\sin(\delta_k x)}{\delta_k}-x\right)-\frac{\cos(\delta_k x)-1}{\delta_k}\cos(2k+1)\pi x\right\|\leq$$

$$\leq \left\| \frac{\sin(\delta_k x)}{\delta_k} - x \right\| + \left\| \frac{1-\cos(\delta_k x)}{\delta_k} \right\| =$$

$$\left\| \frac{1}{\delta_k} \left(\delta_k x - \frac{\delta_k^3 x^3}{3!} + O(\delta_k^5 x^5) \right) - x \right\| + \left\| \frac{1}{\delta_k} \left(1 - 1 + \frac{\delta_k^2 x^2}{2!} - O(\delta_k^4 x^4) \right) \right\| =$$

$$= \left\| x - x - \delta_k^2 \frac{x^3}{3!} + O(\delta_k^4 x^5) \right\| + \left\| \delta_k \frac{x^2}{2!} - O(\delta_k^3 x^4) \right\| \leq O(\delta_k).$$

Taking into account the asymptotics of $\delta_k = O\left(\frac{1}{k}\right)$, we obtain

$$\sum_{k=1}^{\infty} \|\varphi_k - y_k\|^2 = \sum_{k=1}^{\infty} \|\varphi_k^{(2)} - y_k^{(2)}\|^2 \leq C \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

that is the system of functions $\{y_k^{(1)}(x), y_k^{(2)}(x)\}$ and $\{\varphi_k^{(1)}(x), \varphi_k^{(2)}(x)\}$ are quadratically close.

Thus, we have proved

Theorem. *The system of eigenfunctions (4) of the spectral problem (1)-(2) does not form an unconditional basis in $L_2(0,1)$. The auxiliary system of functions (5) forms an unconditional basis in $L_2(0,1)$.*

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2020 Mathematics Subject Classification: 35M10, 35M20

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Frankl-Type Nonlocal Boundary Value Problem for one Parabolic-Hyperbolic Equation

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The talk discusses a new type of boundary value problem involving an equation of mixed type, specifically one that is parabolic-hyperbolic and belongs to the first kind due to the absence of characteristic lines. The equation under consideration is of mixed type, meaning it combines characteristics of both parabolic and hyperbolic equations. Such equations are characterized by different types of behavior in different regions of the domain. This problem involves nonlocal boundary conditions. Nonlocal conditions are those that involve relationships between points on the boundaries of different parts of the domain. In this case, it appears to relate points on the boundaries of the parabolic and hyperbolic parts of the domain. The problem discussed in the talk is a generalization of problems of Frankl type. Frankl type problems are a specific class of mathematical problems that typically involve boundary value problems for partial differential equations. Unlike previous publications, which often considered nonlocal problems in rectangular domains, this talk's problem formulation involves a domain where the hyperbolic part coincides with a characteristic triangle. The talk concludes by claiming the unique strong solvability of the formulated problem. This means that there exists a unique solution that is well-behaved and satisfies the given conditions. In summary, the talk introduces and discusses a novel boundary value problem involving a mixed-type equation with nonlocal boundary conditions.

Let $\Omega \subset R^2$ be a finite domain bounded for $y > 0$ by the segments AA_0, A_0B_0, B_0B, $A = (0,1)$, $B_0 = (1,1)$, $B = (1,0)$ and for $y < 0$ by the characteristics $AC:\ x + y = 0$ and $BC:\ x - y = 1$ of an equation of the mixed parabolic-hyperbolic type

$$Lu = \left\{ \begin{array}{c} u_x - u_{yy}, y > 0 \\ \\ u_{xx} - u_{yy}, y < 0 \end{array} \right\} = f(x,y). \tag{1}$$

This is the equation of the mixed type. The equation refers to the first kind because the line of change of type $y = 0$ is not a characteristic of the equation.

By $W_2^l(\Omega) = H^l(\Omega)$ we denote the space of S.L. Sobolev with the scalar product $(\cdot,\ \cdot)_l$ and the norm $\|\cdot\|_l$, $W_2^0(\Omega) = L_2(\Omega)$; $\Omega_1{=}\Omega \bigcap \{y > 0\}$, $\Omega_2{=}\Omega \bigcap \{y < 0\}$.

In Ω consider the following nonlocal boundary value problem being the generalization of an analogue of the Frankl problem for the parabolic-hyperbolic equation (1).

Problem F. *Find a solution to Eq. (1) satisfying classical boundary conditions*

$$u|_{AA_0} = 0, \quad u_y|_{A_0B_0} = 0 \tag{2}$$

and a nonlocal boundary condition

$$\beta u(\theta(t)) = (1+\alpha)u(\theta_0(t)) + (1-\alpha)u(\theta_1(t)), \quad 0 \le t \le 1, \tag{3}$$

where $\theta(t) = (t,1)$, $\theta_0(t) = \left(\frac{t}{2}, -\frac{t}{2}\right)$, $\theta_1(t) = \left(\frac{t+1}{2}, \frac{t-1}{2}\right)$; α, β *are arbitrary numbers.*

It is easy to see that $\theta(t) \in A_0B_0$, $\theta_0(t) \in AC$, $\theta_1(t) \in BC$. Therefore the new nonlocal boundary condition (3) binds with each other values of the sought-for solution on the parabolic part of the boundary A_0B_0 and on the hyperbolic parts of the boundary of the domain (at the characteristics AC and BC).

Theorem. *Let $\beta \neq 0$. Then for any function $f \in L_2(\Omega)$ there exists a unique strong solution $u(x,y)$ to the problem F. This strong solution belongs to the class $H^1(\Omega) \cap H^{1,2}_{x,y}(\Omega_1) \cap C(\overline{\Omega})$, and satisfies the inequality*

$$\|u\|_1 \leq C\,\|f\|_0.$$

Note that in the special case when $\beta = 1$, the problem F was considered in paper of G.Dildabek [1].

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On the Regularity of the Solution of a Hyperbolic Equation

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Let Ω be a bounded domain in R^n with a sufficiently smooth boundary $\partial\Omega$. We discuss questions about the regularity of the solution the initial-boundary problem

$$\partial_t(t^\beta \partial_t u) - \Delta u = f \text{ in } Q = \Omega \times (0, T), \tag{1}$$

$$u = 0 \text{ on } \Sigma = \partial\Omega \times (0, T), \tag{2}$$

$$u(x, 0) = 0, \ \lim_{t\to+0} t^\beta \partial_t u(x, t) = 0 \text{ in } \Omega, \tag{3}$$

which was studied in the dissertation of N. Kaharman [1]. In particular, they established the following result:

Theorem 1. *Let $\beta \in [0, 1),\ f \in L^2(Q),\ (-\Delta)^{1-\nu} f \in L^2(Q)$. Then problem (1)–(3) is uniquely solvable, and there is an a priori estimate*

$$\|u\|^2_{W^{2,2}_{2,t^\beta}(Q)} \equiv \|t^\beta \partial_t u\|^2_{W^1_2(0,T;\, L^2(\Omega))} + \|u\|^2_{L^2(0,T;W^2_2(\Omega)\cap \overset{\circ}{W}{}^1_2(\Omega))}$$

$$\leq C\left[\|f\|^2_{L^2(Q)} + \|(-\Delta)^{1-\nu} f\|^2_{L^2(Q)}\right], \ \text{where } \nu = \frac{1-\beta}{2-\beta}, \tag{4}$$

i.e. the parameter ν changes within the half-open segment: $\nu \in (0, 1/2]$.

Remark 1. If $\beta = 0$, then equation (1) does not degenerate. However, as the degeneracy of equation (1) "increases" (that is, when parameter β increases from 0 to 1), the requirement for the smoothness of function f on the right side of equation (1) also increases accordingly.

According to Theorem 1 and the results of [2, 3], we have

Theorem 2. *Let $\beta = 0$ and one of the following conditions is met*

$$f \in L^2(Q), \ \partial_t f \in L^2(Q), \tag{4}$$

or

$$f \in L^2(Q), \ (-\Delta)^{1/2} f \in L^2(Q). \tag{5}$$

Then problem (1)–(3) (with a non-degenerate equation) is uniquely solvable in the space

$$u(t) \in W^{2,2}_{2,1}(Q) \equiv \left\{ v(t)|\, v(t) \in L^2(0, T; W^2_2(\Omega)\cap \overset{\circ}{W}{}^1_2(\Omega)), \ \partial_t^2 v(t) \in L^2(Q) \right\},$$

and we have the corresponding a priori estimate

$$\|u\|^2_{W^{2,2}_{2,1}(Q)} \leq C\left[\|f\|^2_{L^2(Q)} + \|\partial_t f\|^2_{L^2(Q)}\right], \tag{6}$$

or

$$\|u\|^2_{W^{2,2}_{2,1}(Q)} \leq C\left[\|f\|^2_{L^2(Q)} + \|(-\Delta)^{1/2} f\|^2_{L^2(Q)}\right]. \tag{7}$$

Note that, in part, the answers to these questions can be obtained using the theory developed in [2, 4]. We present one of the results following from ([2], chapter 5, Theorem 8.1, Remark 8.1 and Proposition 8.1).

Theorem 3. *Let the function f be given by the conditions*

$$f \in L^2(0, T; W^2_2(\Omega)), \ \partial_t^3 f \in L^2(Q), \tag{8}$$

$$f(x, 0) = \partial_t f(x, 0) = \partial_t^2 f(x, 0) = 0. \tag{9}$$

Then the solution $u(x,t)$ of problem (1)–(3) satisfies the inclusions

$$u \in W^{4,4}_{2,1}(Q) \Leftrightarrow u,\ \partial_t u \in L^2(0,T; W^4_2(\Omega) \cap \overset{\circ}{W}{}^1_2(\Omega)),\ \ \partial_t^4 u \in L^2(Q).$$

The following questions arise:

Question 1. What can be said about the regularity of the solution of problem (1)–(3) for $\beta = 1 \cup (1,2) \cup 2$?

Question 2. Is it possible to formulate the conditions of Theorem 1 in terms of the smoothness of the function $f(x,t)$ with respect to the variable t, as it was done in Theorem 2 for the case $\beta = 0$?

Question 3. Is it possible to formulate conditions (8)–(9) of Theorem 3 in terms of the smoothness of the function $f(x,t)$ with respect to the variable x?

In our report, we provide answers to these and other related questions.

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Inequalities between Mixed Moduli of Smoothness

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Let $L_{p_1p_2}, 1 \le p_i \le \infty, i = 1,2$ be the set of measurable functions of two variables $f(x_1,x_2)$, 2π - periodic in each variable, for which $\|f\|_{p_1p_2} = ||\{||f||_{p_1}\}||_{p_2} < \infty$, where

$$\|f\|_{p_i} = \left(\int_0^{2\pi} |f|^{p_i} dx_i\right)^{\frac{1}{p_i}}, \text{ if } 1 \le p_i < \infty,$$

$$\|f\|_{p_i} = \sup_{0 \le x_i \le 2\pi} |f|, \text{ if } p_i = \infty.$$

Let $L^0_{p_1p_2}$ be the space of functions $f \in L_{p_1p_2}$ such that $\int_0^{2\pi} f(x_1,x_2)dx_1 = 0$ for almost all x_2 and $\int_0^{2\pi} f(x_1,x_2)dx_2 = 0$ for almost all x_1.

For the function $f \in L_{p_1p_2}$, we define the fractional differences of positive order α_1 and α_2 with steps h_1 and h_2 respectively, by variables x_1 and x_2 as follows:

$$\Delta^{\alpha_1}_{h_1}(f) = \sum_{\nu_1=0}^{\infty} (-1)^{\nu_1} \tbinom{\alpha_1}{\nu_1} f(x_1 + (\alpha_1 - \nu_1)h_1, x_2),$$

$$\Delta^{\alpha_2}_{h_2}(f) = \sum_{\nu_2=0}^{\infty} (-1)^{\nu_2} \tbinom{\alpha_2}{\nu_2} f(x_1, x_2 + (\alpha_2 - \nu_2)h_2),$$

where $\binom{\alpha}{\nu} = 1$ for $\nu = 0$, $\binom{\alpha}{\nu} = \alpha$ for $\nu = 1$, $\binom{\alpha}{\nu} = \frac{\alpha(\alpha-1)\dots(\alpha-\nu+1)}{\nu!}$ for $\nu \ge 2$.

Denote ([3]) by $\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2}$ the mixed modulus of smoothness of positive orders α_1 and α_2, respectively, in the variables x_1 and x_2 of a function $f \in L_{p_1p_2}$, that is,

$$\omega_{\alpha_1,\alpha_2}(f,\delta_1,\delta_2)_{p_1p_2} = \sup_{|h_i| \le \delta_i, i=1,2} ||\Delta^{\alpha_1}_{h_1}\left(\Delta^{\alpha_2}_{h_2}(f)\right)||_{p_1p_2}.$$

The following (p,q)-inequality between moduli of smoothness in different metrics, nowadays called sharp Ulyanov type inequalities, is known (see [5]):

$$\omega_1(f,\delta)^{(1)}_q \ll \left(\int_0^{\delta} \left(t^{-\theta}\omega_1(f,t)^{(1)}_p\right)^{q^*} \frac{dt}{t}\right)^{\frac{1}{q^*}},$$

where $1 \le p < q \le \infty, \theta = \frac{1}{p} - \frac{1}{q}$. For the Lebesgue spaces, Ulyanov type inequalities in the one-dimensional case have been studied for the moduli of smoothness of any positive order by many authors, in particular, Ulyanov [5], V.I. Kolyada [1], Yu. Kolomoitsev, S. Tikhonov [2], B. Simonov [4].

Theorem 1. *Let $f \in L^0_{p_1p_2}$, where $1 < p_1 < q_1 < \infty$ or $1 = p_1 < q_1 = \infty$ and $1 < p_2 < q_2 < \infty$ or $1 = p_2 < q_2 = \infty$. Let for $\alpha_i > 0, \delta_i \in (0,1), i = 1,2,$ we have*

$$\omega_{\alpha_1,\alpha_2}\big(f,\delta_1,\delta_2\big)_{q_1q_2} \ll$$

$$\Big(\int_0^{\delta_2}\Big(\int_0^{\delta_1}\Big(t_1^{-\frac{1}{p_1}+\frac{1}{q_1}} t_2^{-\frac{1}{p_2}+\frac{1}{q_2}} \omega_{\alpha_1+\frac{1}{p_1}-\frac{1}{q_1},\alpha_2+\frac{1}{p_2}-\frac{1}{q_2}}(f,t_1,t_2)_{p_1p_2}\Big)^{q_1^*}\frac{dt_1}{t_1}\Big)^{\frac{q_2^*}{q_1^*}}\frac{dt_2}{t_2}\Big)^{\frac{1}{q_2^*}}.$$

Theorem 2. *Let $f \in L^0_{p_1p_2}$, where $1 < p_1 < q_1 < \infty$ or $1 = p_1 < q_1 = \infty$ and $1 < p_2 < q_2 < \infty$ or $1 = p_2 < q_2 = \infty$. Let for $\alpha_i > 0, \rho_i \ge 0, \delta_i \in (0,1), i = 1,2.$ we have*

$$\omega_{\alpha_1,\alpha_2}\big(f^{(\rho_1,\rho_2)},\delta_1,\delta_2\big)_{q_1q_2} \ll$$

$$\Big(\int\limits_0^{\delta_2}\Big(\int\limits_0^{\delta_1}\Big(t_1^{-\rho_1-\frac{1}{p_1}+\frac{1}{q_1}}t_2^{-\rho_2-\frac{1}{p_2}+\frac{1}{q_2}}\omega_{\alpha_1+\rho_1+\frac{1}{p_1}-\frac{1}{q_1},\alpha_2+\rho_2+\frac{1}{p_2}-\frac{1}{q_2}}(f,t_1,t_2)_{p_1p_2}\Big)^{q_1^*}\frac{dt_1}{t_1}\Big)^{\frac{q_2^*}{q_1^*}}\frac{dt_2}{t_2}\Big)^{\frac{1}{q_2^*}}.$$

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On one Method for Solving a Boundary Value Problem for a Nonlinear Loaded Ordinary Differential Equation

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In the domain $\bar{\Omega} = [0, \omega] \times [0, T]$ we considered semi-periodic boundary value problem for a semi-linear loaded hyperbolic equations:

$$\frac{\partial^2 u}{\partial x \partial t} = A(x,t)\frac{\partial u(x,t)}{\partial x} + A_0(x,t)\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=x_0} + f\Big(x, t, u(x,t), \frac{\partial u(x,t)}{\partial t}\Big), \quad (1)$$

$$u(x,0) = u(x,T), \qquad x \in [0,\omega], \quad (2)$$

$$u(0,t) = \psi(t), \qquad t \in [0,T], \quad (3)$$

where $f : \overline{\Omega} \times R^2 \to R$, continuous on $\bar{\Omega}$, $\psi(t)-$ continuously differentiable on $[0,T]$ and satisfying the condition $\psi(0) = \psi(T)$ functions.

Function $u(x,t) \in C(\overline{\Omega})$, having partial derivatives $\dfrac{\partial u(x,t)}{\partial x} \in C(\overline{\Omega})$, $\dfrac{\partial u(x_0,t)}{\partial x} \in C(\overline{\Omega})$, $\dfrac{\partial u(x,t)}{\partial t} \in C(\overline{\Omega})$, $\dfrac{\partial^2 u(x,t)}{\partial x \partial t} \in C(\overline{\Omega})$, is called a classical solution of problem (1)-(3), if it satisfies equation (1) for all $(x,t) \in \overline{\Omega}$ and the boundary conditions (2)-(3).

We introduce new unknown functions $v(x,t) = \dfrac{\partial u(x,t)}{\partial x}$, $w(x,t) = \dfrac{\partial u(x,t)}{\partial t}$, and we must take into account that $v(x_0,t) = \left.\dfrac{\partial u}{\partial x}\right|_{x=x_0}$, and problem (1)-(3) is reduced to the following equivalent problem:

$$\frac{\partial v}{\partial t} = A(x,t)v(x,t) + A_0(x,t)v(x_0,t) + f(x,t,u(x,t),w(x,t)), \quad (4)$$

$$v(x,0) = v(x,T), \quad x \in [0,\omega], \quad (5)$$

$$u(x,t) = \psi(t) + \int_0^x v(\xi,t)d\xi, \;\; w(x,t) = \dot{\psi}(t) + \int_0^x v_t(\xi,t)d\xi \quad (6)$$

To find a solution to the boundary value problem (4)-(6), we use a modification of the Euler polyline method [1]. Using this method, we obtained conditions for the existence of boundary value problem (1)-(3). Boundary value problems for loaded differential equations have been studied by many authors [2-7].

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Keywords: loaded differential equation, impulse effect, parametrization method, algorithm.

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A New Computational Approach for Solving Problem for Impulsive Differential Equations with Loadings

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We consider the following linear two-point boundary value problem for systems of essentially loaded differential equations with impulse effect:

$$\frac{dx}{dt} = A_0(t)x + \sum_{i=1}^{m} M_i(t) \lim_{t\to\theta_i+0} \dot{x}(t) + \sum_{i=1}^{m} A_i(t) \lim_{t\to\theta_i+0} x(t) + f(t), \quad t \in (0,T), \tag{1}$$

$$B_0x(0) + C_0x(T) = d, \quad d \in R^n, \quad x \in R^n, \tag{2}$$

$$B_i \lim_{t\to\theta_i-0} x(t) - C_i \lim_{t\to\theta_i+0} x(t) = \varphi_i, \quad \varphi_i \in R^n, \quad i = \overline{1,m}. \tag{3}$$

Here $(n\times n)$-matrices $A_j(t)$, $(j = \overline{0,m})$, $M_i(t)$, $(i = \overline{1,m})$, and n-vector-function $f(t)$ are piecewise continuous on $[0,T]$ with possible discontinuities of the first kind at the points $t = \theta_i$, $(i = \overline{1,m})$. B_j and C_j, $(j = \overline{0,m})$ are constant $(n \times n)$ - matrices, and φ_i, $(i = \overline{1,m})$ are constant n vector functions, $0 = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{m-1} < \theta_m < \theta_{m+1} = T$.

A solution to problem (1) - (3) is a piecewise continuously differentiable vector function $x(t)$ on $[0,T]$ which satisfies the system of essentially loaded differential equations (1) on $[0,T]$ except the points $t = \theta_i$, $(i = \overline{1,m})$, the boundary condition (2), and conditions of impulse effects at the fixed time points (3).

Impulsive differential equations, that is, differential equations involving impulse effects, appear as a natural description of observed evolution phenomena of several real world problems [1]. Problems for impulsive differential equations with loadings and methods for finding their solutions are considered in [2].

Present work considers a two-point boundary value problem for the system of loaded ordinary differential equations with impulse effect (1) - (3). Loading points in differential equation are also the points of impulse effect. According to Dzhumabaev parametrization method [3] the interval is partitioned into parts, then the values of solution at the initial points of subintervals are introduced as additional parameters. By the matrices of the loaded summands, boundary condition, and conditions of impulse effect we construct a system of linear algebraic equations with respect to parameters. Coefficients and right-hand side of this system are determined by the solutions of Cauchy problems for linear ordinary differential equations. A numerical algorithm is offered for solving the problem (1) - (3).

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Keywords: loaded differential equation, impulse effect, parametrization method, algorithm.

2020 Mathematics Subject Classification: 34K10, 34K45, 45J05

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Weighted Sobolev Type Inequalities and Identities

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Inspired by the work of Badiale-Tarantello [1], in this talk we discuss sharp remainder formulae for the cylindrical extensions of the improved Hardy inequalities with weights. For more general p we obtain cylindrical improved L^p-Hardy identities with weights for all real-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$, while in L^2 case we have them for any complex-valued function $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. Moreover, we show cylindrical L^p-Hardy inequalities with weights for all complex-valued functions $f \in C_0^\infty(\mathbb{R}^n \backslash \{x' = 0\})$. As applications, we establish improved Caffarelli-Kohn-Nirenberg type inequalities with remainder terms. In addition, we also discuss the results in the setting of homogeneous Lie groups.

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Keywords: Sobolev type identity, cylindrical extension, sharp remainder formula, Caffarelli-Kohn-Nirenberg inequality, anisotropic Euclidean space.

2020 Mathematics Subject Classification: 35Q79, 35K05, 35K20

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The Criterion of Minimality of the Mixed Type Gellerstedt Equation

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Minimal operators generated by overdetermined boundary conditions for differential equations are extremely important when describing regular boundary value problems for differential equations. In addition, for inverse problems of mathematical physics arising from applications, when determining unknown data, it is necessary to study problems with overdetermined boundary conditions, including the boundary conditions of minimal operators. Thus, the study of minimal differential operators is of both theoretical and applied interest. In this paper, a criterion for the regular solvability of the differential operator generated by the overdetermined Cauchy problem for the Gellerstedt equation is established. The proof is based on the Gellerstedt potential, the properties of solutions to the Goursat problem in the characteristic triangle, and the properties of special functions. It should be noted that differential operators of mixed type have numerous applications in transonic gas dynamics, the theory of infinitesimal surface bends, the instantaneous theory of shells with variable sign curvature, magnetodynamics and hydrodynamics.

Let $\Omega \subset R^2$ be a domain, bounded by segment $AB : y = 0,\ 0 < x < 1$, and by characteristics $AC:\ x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0$ and $BC:\ x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 1$ of the Gellerstedt equation

$$Lu \equiv -(-y)^m u_{xx} + u_{yy} = f(x,y). \tag{1}$$

The overdetermined Cauchy problem is considered: to find a regular solution of (1) equation in the Ω domain, satisfying the conditions:

$$u|_{y=0} = 0, \quad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0, \tag{2}$$

$$u\big|_{AC:\ x-\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=0} = 0, \quad u\big|_{BC:\ x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}=1} = 0. \tag{3}$$

Theorem 1. *The minimal operator ((1), (2), (3) problem) is invertible in $L_2(\Omega)$ if and only if the following conditions are met*

$$\int_0^\xi d\xi_1 \int_1^\xi \frac{(\eta_1-\xi_1)^{2\beta}}{(\xi-\xi_1)^\beta(\eta_1-\xi)^\beta} \cdot f_1(\xi_1,\eta_1)\, d\eta_1 = 0, \tag{4}$$

$$\int_0^\xi d\xi_1 \int_1^\xi \frac{\eta_1-\xi_1}{(\xi-\xi_1)^{1-\beta}(\eta_1-\xi)^{1-\beta}} \cdot f_1(\xi_1,\eta_1)\, d\eta_1 = 0, \tag{5}$$

where the function f_1 is determined by a given function $f(x,y)$ from (1). In this case, the solution of overdetermined Cauchy problem is representable by the formula:

$$u(\xi,\eta) = \int_0^\xi d\xi_1 \int_1^\eta R(\xi,\eta,\xi_1,\eta_1) \cdot f_1(\xi_1,\eta_1)\, d\eta_1, \tag{6}$$

where $R(\xi,\eta,\xi_1,\eta_1)$ is the Riemann function of the Goursat problem.

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Keywords: minimal differential operator, Gellerstedt equation, criterion, boundary condition, hypergeometric function.

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Weighted Hardy-Type Inequalities for Monotone Functions

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Let $I = (0, \infty)$, $1 < p, q < \infty$ and $p' = \frac{p}{p-1}$. Suppose that v, u and $v^{1-p'}$ are positive functions locally integrable on I.

We consider the following Hardy-type inequality

$$\left(\int_0^\infty u(x) \left| \int_0^x K(x,t) f(t) dt \right|^q dx \right)^{\frac{1}{q}} \le C \left(\int_0^\infty v(x) |f(x)|^p dx \right)^{\frac{1}{p}}, \tag{1}$$

for all functions $f \in L_{p,v}(I)$, where $L_{p,v}(I)$ is the weighted Lebesgue space such that $\|f\|_{p,v} = \left(\int_0^\infty v(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty$. Here

$$Kf(x) = \int_0^x K(x,t) f(t) dt, \; x > 0, \tag{2}$$

is an integral operator with a non-negative kernel $K(x,t)$.

At present, there are many works devoted to Hardy-type inequalities with iterated operators. Motivated by important applications, all these generalizations of the Hardy inequality are studied not only on the cone of non-negative functions but also on the cone of monotone functions. This work discusses new Hardy-type inequalities for operators with kernels that satisfy less restrictive conditions than those considered earlier. The presented inequalities have already been characterized for non-negative functions. In this work, we continue this study but for monotone functions. To achieve the aim, we use the famous Sawyer duality principle, which gives an equivalence between the Hardy inequality for monotone functions and some inequality for non-negative functions.

Heat Polynomials Method for Solving Inverse Stefan Type Problems

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In a series of studies [1-4], we have developed the approximate heat polynomials method (HPM) for solving various inverse Stefan-type problems. The HPM, or polynomial Trefftz method [5], was initially described by Appell in 1892 in his paper [6], with further development in subsequent works [7-11].

The essence of this method lies in constructing an approximate solution that satisfies the heat conduction equation in the form of a linear combination of heat polynomials. The unknown coefficients are determined through the minimization of residuals, which can be achieved through various methods such as the variational method, the least squares method, or other techniques.

We have investigated the convergence and stability of the HPM through numerous numerical examples. A novel aspect of this work is the development of the collocation heat polynomials method for solving inverse Stefan-type problems.

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Keywords: inverse Stefan problems, approximate solution, heat polynomials method, moving boundary.

2010 Mathematics Subject Classification: 80A22, 80A23, 80M30, 35C11.

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Nonlinear Stefan Problem with a Heat Source Arising in Closed Electrical Contacts

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This research introduces a mathematical framework that characterizes the heat dynamics occurring during the vaporization of metal in electrical contacts. The study establishes an arc erosion model for micro-asperities when in a liquid state, employing a one-phase Stefan problem approach. The investigation addresses two scenarios involving the impact of Joule heating on a generalized heat equation. By leveraging the similarity principle, the problems are simplified into ordinary differential equations for solution. The existence and uniqueness of these solutions are demonstrated using fixed point theory within a Banach space. Furthermore, a solution is presented for the specific scenario where thermal coefficients remain constant.

The mathematical formulation describing the behavior of the liquid phase involves a generalized heat equation that accommodates a bar with a varying cross-section:

$$c(\theta_1)\gamma(\theta_1)\frac{\partial\theta_1}{\partial t} = \frac{1}{z^\nu}\frac{\partial}{\partial z}\left[\lambda(\theta_1)z^\nu\frac{\partial\theta_1}{\partial z}\right] + \rho(\theta_1)j^2, \quad z_1(t) < z < z_2(t), \quad 0 < t < t_a, \tag{1}$$

Within this context, $\theta_1(z,t)$ signifies the temperature in the liquid phase. Parameters such as $c(\theta_1)$, $\gamma(\theta_1)$, $\lambda(\theta_1)$, and $\rho(\theta_1)$ are specific heat, density, heat conductivity, and electrical resistivity, respectively, all of which depend on temperature and consequently introduce nonlinearity into the problem. The scaling symmetry of micro-asperity is captured by $\nu > 1$, t_a represents the duration of arcing, and j symbolizes the current density. The boundary conditions are as follows:

$$\theta_1(z_1(t),t) = \theta_b, \tag{2}$$

$$\theta_1(z_2(t),t) = \theta_m. \tag{3}$$

Here, θ_b denotes the boiling temperature, and θ_m signifies the melting temperature. The position of the interface where the temperature reaches the melting point is identified as $z = z_2(t)$ and can be determined by applying the Stefan condition:

$$-\lambda(\theta_1(z_2(t),t))\frac{\partial\theta_1(z_2(t),t)}{\partial z} = l_m\gamma_m\frac{dz_2}{dt}, \quad t > 0. \tag{4}$$

The latent heat and density at the melting temperature are represented as l_m and γ_m, respectively. The initial temperature field condition in this region is specified as:

$$\theta_1(z,0) = \theta_m, \quad z_1(0) = z_2(0) = 0. \tag{5}$$

The Joule heating term in the equation (1) can be described in the form

$$\rho(\theta_1)j^2 = \rho(\theta_1)\frac{I_0^2\sin^2(\omega t)}{\pi^2 z^{2\nu}}$$

where

$$\sin^2(\omega t) \approx kt^{\nu-1}, \;\; k = \frac{\sin^2(\omega t_a)}{t_a^{\nu-1}}.$$

We will examine the one more alternative problem that involves substituting condition (2) with a heat flux condition described by equation below:

$$-\lambda(\theta_1(z_1(t),t))\frac{\partial\theta_1(z_1(t),t)}{\partial z} = \frac{P_0 e^{-z_0^2}}{2a\sqrt{\pi t}}. \tag{6}$$

In this context, $P_0e^{-z_0^2}/(2a\sqrt{\pi t})$ represents a specified heat flux characterized by a positive power value P_0.

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Keywords: non-classical Stefan problem, generalized heat equation, Joule heat source, nonlinear thermal coefficients, fixed point theory.

2020 Mathematics Subject Classification: 35C11, 80A23, 80A22

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Stein-Weiss-Adams Inequality on Morrey Space

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We establish Adams type Stein-Weiss inequality on global Morrey spaces on general homogeneous groups. Special properties of homogeneous norms and some boundedness results on global Morrey spaces play key roles in our proofs. As consequence, we obtain fractional Hardy, Hardy-Sobolev, Rellich and Gagliardo-Nirenberg inequalities on Morrey spaces on stratified groups. While the results are obtained in the setting of general homogeneous groups, they are new already for the Euclidean space $\mathbb{R}^N$. This work was jointly with M.A. Ragusa, M. Ruzhansky and D.Suragan and published in [1].

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Keywords: global Morrey space, Stein-Weiss inequality.

2020 Mathematics Subject Classification: 22E30, 43A80

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Solution of Initial-Boundary Value Problems for the Heat Equation with Discontinuous Coefficients

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The initial-boundary value problem for the heat equation with discontinuous coefficients

$$\frac{\partial u_i}{\partial t} = k_i^2 \frac{\partial^2 u_i}{\partial x^2}, \tag{1}$$

is considered in the domain $\Omega = \cup\Omega_i,\ \Omega_i = \{(x,t),\, l_{i-1} < x < l_i,\, 0 < t < T\}$ $(i = 1, 2, 3)$, with initial conditions

$$u(x,0) = \varphi(x),\ \ l_0 < x < l_3, \tag{2}$$

boundary conditions of the form

$$\begin{cases} \alpha_1 \frac{\partial u_1(l_0,t)}{\partial x} + \beta_1 u_1(l_0,t) = 0, \\ \alpha_2 \frac{\partial u_3(l_3,t)}{\partial x} + \beta_2 u_3(l_3,t) = 0, \end{cases} \quad 0 \le t \le T, \tag{3}$$

and pairing conditions

$$u(l_i - 0, t) = u(l_i + 0, t),\ \ 0 \le t \le T\ \ (i = 1, 2), \tag{4}$$

$$k_i \frac{\partial u_i(l_i - 0, t)}{\partial x} = k_{i+1} \frac{\partial u_{i+1}(l_i + 0, t)}{\partial x},\ \ 0 \le t \le T,\ \ (i = 1, 2). \tag{5}$$

In the case without discontinuity, the spectral theory of these problems is constructed almost completely. In [1], the heat equation with a discontinuous coefficient was considered under Sturm-type boundary conditions (separated boundary conditions), eigenvalues and eigenfunctions were found, and various special cases were investigated.

In this paper, the spectral questions of problem (1)-(5) are investigated. The eigenvalues and eigenfunctions are found, and the existence and uniqueness theorem for the classical solution is proved.

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Keywords: Heat equation, discontinuous coefficients, eigenvalues, eigenfunctions, spectral theory.

2020 Mathematics Subject Classification: 35K05, 35K15

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On the Fractional Neutral Levin-Nohel Integro-Differential Equations

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This work deals with the nonlinear neutral Levin-Nohel integro-differential equation with Caputo fractional derivative and variable delays. The Ulam-Hyers-Rassias stability, Ulam-Hyers stability, semi-Ulam-Hyers-Rassias stability are studied. The existence and uniqueness of solutions are established by using Krasnoselskii's fixed point theorem and contraction mapping principle.

Keywords: Levin-Nohel integro-differential equation, Caputo fractional derivative, existence of solution, uniqueness of solution, stability of solution, Krasnoselskii's fixed point theorem.
2020 Mathematics Subject Classification: 35Q79, 35K05, 35K20

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Lieb–Thirring Inequalities on Manifolds with Negative Curvature

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We prove Lieb–Thirring inequalities on manifolds with negative constant curvature. The discrete spectrum appears below the continuous spectrum. In order to prove the main statement we use the result regarding Lieb–Thirring inequalities for Schrödinger operators with operator-valued potentials. As an application we obtain a Pólya type inequality with not a sharp constant

On Hardy-Littlewood theorem

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This report is devoted to relation between integrability properties of functions and summability properties of their Fourier coefficients. In particular, we prove Hardy-Littlewood type theorem.

For functions $f(x) \in L_1([0,1])$ with Fourier series $\sum\limits_{k=-\infty}^{\infty} c_k e^{2\pi k x}$, where $\{c_k\}_{k=0}^{\infty}$ and $\{c_k\}_{-\infty}^{0}$ are nonincreasing sequences the Hardy-Littlewood theorem holds i.e., there exist $C_1, C_2 > 0$ such that

$$C_1\|f\|_{L_p} \leq \left(\sum_{k=-\infty}^{\infty} (|k|+1)^{p-2} c_k^p\right)^{\frac{1}{p}} \leq C_2\|f\|_{L_p}. \tag{1}$$

Equivalence (1) has a lot of generalizations, see for instance, [1], [2], [3] and references therein. In this work, we get a new generalizations of Hardy-Littlewood's theorem. In particular, we obtain the equivalence (1) for functions with Fourier coefficients $\{c_k\}_{k=-\infty}^{\infty}$ such that, for any $k \geq 0$ and some $C > 0$, the following inequality

$$\sum_{[2^{k-1}]\leq|m|<[2^k]} |a_m - a_{m+1}| \leq C \sup_{r\in\mathbb{N}} \min(1, 2^{r-k}) \sup_{2^{r-1}\leq|m|<2^r} \frac{1}{|m|}\left|\sum_{j=0}^{m} a_j\right|$$

holds, where $[x]$ is an integer part of number x.

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Keywords: trigonometric series, general monotonicity, Hardy-Littlewood theorem.

2020 Mathematics Subject Classification: 42A16, 42A32

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Mathematical Modeling of Fluid Filtration Processes with Consideration of Mass Transfer Processes

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The study is dedicated to approximate methods for solving the problem of non-isothermal filtration of immiscible fluids. The qualitative properties of solutions have been investigated. The obtained results were utilized in the development of digital technologies for oil and gas fields.

The problem formulation is considered through the Maskeet-Leverett temperature model.

The qualitative properties of solutions to problems (1) - (5) have been investigated, and computational algorithms have been developed for numerical implementation on a computer. Based on these algorithms, test cases with real data from a specific oil field in the western region of the Republic of Kazakhstan have been conducted.

$$\frac{\partial\Theta}{\partial t} = div(\lambda(x,s,\Theta)\nabla\Theta - \vec{v}\Theta), \tag{1}$$

$$m\frac{\partial s}{\partial t} = div(Ka_0(a_1\nabla\sigma - a_2\nabla\Theta + \vec{f_1}) - b_1\vec{v} = div\vec{v_1}, \tag{2}$$

$$div\vec{v} = div(K(\nabla p + a_3\nabla\Theta + \vec{f_2})), \tag{3}$$

$$\sigma = \frac{s - s_*(\Theta)}{s^*(\Theta) - s_*(\Theta)}$$

on $s_* \leq s(x,t) \leq s^*; \sigma = 0 \; on \;\; s < s_*(\Theta), \sigma = 1 \;\; on \;\; s > s^*(\Theta)$ The last condition defines the function

$$\sigma = \Phi(s,\Theta)$$

at

$$s \in (0,1),$$

where Θ is equal to the temperature of the non-uniform liquid, λ is the thermal conductivity coefficient $s = s_1$ is the saturation of the wetting phase

$$\vec{v} = \vec{v_1} + \vec{v_2}$$

is the average filtration velocity of the mixture $\vec{v_i}$ is the phase filtration velocities.

Additionally, in the considered model, the residual saturations are not constant, denoted as $s_i^0 = s_i^0(\Theta) > s_i^{-0} = const > 0$, i=1,2. These specified properties lead to the following conditions for the saturation $s_i^0, \;\; i = 1,2 \; s(x,t)$ (wetting phase):

$$0 \leq const = \vec{s_*} \leq s_*(\Theta) \leq s(x,t) \leq s^*(\Theta) \leq s^*(\Theta) \leq \vec{s^*} = const \leq 1$$

where $\vec{s_*} = s_1^0(\Theta), \quad \vec{s^*} = 1 - s_2^0(\Theta). \quad K = K(x,\Theta,\sigma)$ the tensor associated with the permeability of a medium.

$a_0 = a_0(s),\, a_i(\sigma,\Theta),\, i = 1,2,3 \quad b_k = b_k(\sigma,\Theta, \quad \vec{f_k} = \vec{f_k}(x,\sigma,\Theta \quad k = 1,2$

$a_0(0) = a_0(1), b_1(0,\Theta) = 0; \inf a_1 > a_0 > 0$

Let $\Omega \in R^3$ be bounded region, the boundary $\partial\Omega$ that is divided into several components depending on the type of boundary conditions:

$$(P,S,\Theta) = (P_0,S_0,\Theta_0, \;\; (x,t) \in \Sigma^1 = \Gamma^1 x(0,T)$$

$$\vec{v_i}\ \vec{n} = b_i\ \psi,\ i = 1, 2; \Theta = \Theta_0(x,t),\ (x,t) \in \Sigma^2 = \Gamma^2 x(0,T)$$
$$\vec{v_i}\ \vec{n} = b_i\ \psi,\ i = 1, 2;$$
$$\lambda\ \frac{\partial \Theta}{\partial n} = \beta\ (\Theta_0 - \Theta), (x,t) \in \Sigma^3 = \Gamma^3 x(0,T)$$

To the boundary conditions (4), it is necessary to add initial conditions:

$$(s, \Theta)_{t=0} = (s_0, \Theta_0)(x, 0),\quad x \in \Omega$$

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Initial-Boundary Value Problem for the Heat Equation under Antiperiodic Boundary Conditions in the Absence of Agreement Between the Initial and Boundary Data

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In the report, the initial-boundary value problem for the heat equation under antiperiodic boundary conditions in the absence of agreement between the initial and boundary data is considered.

It is well known that when studying boundary value problems for parabolic equations in Hölder space, it is necessary to require the fulfillment of conditions for the compatibility of boundary and initial data at the boundary of the domain at $t = 0$. These conditions ensure the continuity of the solution and its derivatives, as well as the boundedness of the Hölder constants of higher derivatives inside the domain. Consistency conditions are functional identities that connect these functions on the boundary of the domain at the initial moment.

Problem P. Find in $\Omega = \{(x,t) : 0 < x < l, 0 < t < T\}$ a solution $u(x,t)$ of the heat equation

$$u_t - k^2 u_x x = f(x,t), \tag{1}$$

satisfying the initial condition

$$u(x,0) = \varphi(x), 0 \leq x \leq l, \tag{2}$$

and antiperiodic boundary conditions

$$\begin{cases} u_x(0,t) + u_x(l,t) = 0, \\ u(0,t) + u(l,t) = 0. \end{cases} \tag{3}$$

It is well known that for the existence of a classical solution, the matching conditions must be satisfied. For example, the zero and first order matching conditions for problem (1)-(3) are

$$A_0 \equiv \varphi(0) + \varphi(l) = 0, \ A_1 \equiv \varphi'(0) + \varphi'(l) = 0.$$

The second-order matching condition arises when we consider solutions of the problem from the class $u \in C^{2,1}_{x,t}(\bar{\Omega})$. For functions from such a class, we can pass to the limit in equation (1) at $t \to 0$ at $x = 0$ and $x = l$. Then we get

$$A_2 \equiv k^2[\varphi''(0) + \varphi''(l)] - [f(0,0) + f(l,0)] = 0.$$

For a problem with Dirichlet boundary conditions, solutions with a mismatch between the boundary and initial data were studied in [1].

In this report, we consider the problem (1)-(3) with nonlocal boundary conditions. Cases where the matching conditions of the zero, first and second orders are not met are considered in the report.

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Keywords: heat equation, initial-boundary value problem, nonlocal boundary conditions, matching conditions.

2020 Mathematics Subject Classification: 35K05, 335K15

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On a Semi-Minimal Degenerate Hyperbolic Operator

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Let $\Omega \subset R^n$ be bounded simply-connected domain with smooth boundary $\partial\Omega$, and denote $D = (0, T) \cap \Omega$.

Let us consider the following equation on the domain D

$$Lu = \frac{\partial}{\partial t}\left(K(t)\frac{\partial}{\partial t}u\right) - \Delta_x u = f(x,t), \tag{1}$$

where $K(t) = t^\beta(1-t)^\beta K_1(t),\ K_1(t) \in C^2[0,T]$.

Denote L_0 as the closure of the differential operator (1) in $L_2(D)$ on a subset of functions $u \in \overset{o}{W}{}_2^2\,(0,T) \cap C^2(\overline{\Omega})$, which satisfies the lateral potential boundary condition

$$-\frac{u(x,t)}{2} - \int\limits_{\partial\Omega}\left(\varepsilon_n(x,\xi)\frac{\partial u(\xi,t)}{\partial n_\xi} - \frac{\partial \varepsilon_n(x,\xi)}{\partial n_\xi}u(\xi,t)\right)dS_\xi = 0, \quad x \in \partial\Omega,\ t \in [0,T]. \tag{2}$$

where $\varepsilon(x,\xi)$ is the fundamental solution of the Laplace equation

$$-\Delta_x\varepsilon(x,\xi) = \delta(x-\xi),\ x,\xi \in \Omega. \tag{3}$$

By using spectral decomposition to the self-adjoint problem

$$-\Delta_x e_m(x) = \lambda_m e_m(x),\ x \in \Omega, \tag{4}$$

$$-\frac{e_m(x)}{2} - \int\limits_{\partial\Omega}\left(\varepsilon_n(x,\xi)\frac{\partial e_m(\xi)}{\partial n_\xi} - \frac{\partial \varepsilon_n(x,\xi)}{\partial n_\xi}e_m(\xi)\right)dS_\xi = 0, \quad x \in \partial\Omega, \tag{5}$$

have found reversibility conditions for the semi-minimal operator L_0^{-1}.

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Weighted Estimates for one Class of Quasilinear Integral Operators with Kernels

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Let $I = (0,\infty)$, $1 < p,r,q < \infty$ and $p' = \frac{p}{p-1}$. Let u, v and w be positive functions locally summable on I such that $u \in L_q^{loc}(I)$ and $v^{-1} \in L_{p'}^{loc}(I)$. Let $\|f\|_{p,v} = \left(\int\limits_0^\infty |v(t)f(t)|^p dt\right)^{\frac{1}{p}}$ for $1 \le p < \infty$.

We consider the weighted inequality

$$\|Rf\|_{q,u} \le C\|f\|_{p,v}, \ f \ge 0, \tag{1}$$

where R is one of the following operators

$$Tf(x) = \left(\int\limits_0^x G(x,t)w(t)\left(\int\limits_t^\infty K(s,t)f(s)ds\right)^r dt\right)^{\frac{1}{r}}, \tag{2}$$

$$Sf(x) = \left(\int\limits_0^x G(x,t)w(t)\left(\int\limits_0^t K(t,s)f(s)ds\right)^r dt\right)^{\frac{1}{r}}, \tag{3}$$

$$T^-f(x) = \left(\int\limits_x^\infty G(t,x)w(t)\left(\int\limits_o^t K(t,s)f(s)ds\right)^r dt\right)^{\frac{1}{r}}, \tag{4}$$

$$S^-f(x) = \left(\int\limits_x^\infty G(t,x)w(t)\left(\int\limits_t^\infty K(s,t)f(s)ds\right)^r dt\right)^{\frac{1}{r}}. \tag{5}$$

The inequality (1) for the operators (2)–(5) with $G(\cdot,\cdot) \equiv K(\cdot,\cdot) \equiv 1$ has been studied, for example, in the works [1] and [2]. The case, when $G(\cdot,\cdot) \equiv 1$ and $K(\cdot,\cdot)$ satisfies the condition $\mathcal{O}$ stating that $K(x,s) \ge 0$ for $x \ge s > 0$ and $K(x,s) \approx K(x,t) + K(t,s)$ for $x \ge t \ge s > 0$, has been considered, for example, in [3].

A more general case, when $G(\cdot,\cdot)$ and $K(\cdot,\cdot)$ satisfy the condition $\mathcal{O}$, has been investigated, for example, in [4] and [5].

Here we present the results for the inequality (1) when $R \equiv T$ and $R \equiv S$. Similar results are obtained for $R \equiv T^-$ and $R \equiv S^-$.

We assume that the functions $K(\cdot,\cdot)$ and $G(\cdot,\cdot)$ belong to the classes $\mathcal{O}_n^\pm$, $n \ge 1$, and $\mathcal{O}_m^\pm, m \ge 1$, respectively. The classes $\mathcal{O}_n^\pm$, $n \ge 1$, were introduced in [6]. These classes are wider than the class $\mathcal{O}$.

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Keywords: Integral operator, Hardy-type inequality, Weight function, Kernel.

2020 Mathematics Subject Classification: 26D10, 26D15

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Navier-Stokes-Voigt Equations Governing Density-Dependent Flows with Vacuum

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Let us consider the following initial-boundary value problem for the Navier-Stokes-Voigt system that governs density-dependent flows (nonhomogeneous flows) of incompressible fluids with elastic properties,

$$\operatorname{div} u = 0 \qquad \text{in} \quad Q_T, \tag{1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) = \rho f - \nabla p + \mu \Delta u + \kappa\, \Delta u_t \qquad \text{in} \quad Q_T, \tag{2}$$

$$\rho_t + \operatorname{div}(\rho u) = 0, \quad \rho \geq 0 \qquad \text{in} \quad Q_T, \tag{3}$$

$$\rho u = \rho_0 u_0, \quad \rho = \rho_0 \qquad \text{in} \quad \{0\} \times \Omega, \tag{4}$$

$$u = 0 \qquad \text{on} \quad \Gamma_T. \tag{5}$$

Here, $\Omega \subset \mathbb{R}^d$ is a bounded domain, $Q_T = (0,T) \times \Omega, 0 < T < \infty$ is a cylinder with lateral $\Gamma_T = (0,T) \times \partial\Omega$. The unknowns of the problem are u, ρ and p, while f, u_0 and ρ_0 are given data. For now, we consider a general space dimension $d \geq 2$, though the real-world applications correspond to the cases $d = 2,\ 3$. In these cases, u denotes the velocity field, ρ accounts for the density, p is the pressure, f stands for the external forces field, while μ and κ are positive constants to the dynamic viscosity and to the relaxation time. We are interested in the case of the initial data u_0 and ρ_0 satisfying

$$\operatorname{div} u_0 = 0 \quad \text{in} \quad \Omega,$$

$$0 \leq \rho_0 \leq M < \infty \quad \text{in} \quad \Omega,$$

for some positive constant M. The case when $0 < m \leq \rho \leq M < \infty$ was considered in [1].

In this talk, we prove the existence and uniqueness of strong solution above possed problem (1)-(5).

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Keywords: Navier-Stokes-Voigt model, incompressible fluids with non-constant density, nonhomogeneous and incompressible fluids, existence.

2020 Mathematics Subject Classification: 35Q35, 76D03, 76A10

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Initial Boundary Value Problems for the Wave Equation with Not Strongly Regular Boundary Conditions

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Initial-boundary value problems for linear equations of hyperbolic type are a fairly well-developed part of the theory of partial differential equations (see, for example, [1-3]). And one of the most developed methods for solving them is the Fourier method, which is also called the method of separation of variables or the method of expansion in terms of eigenfunctions. This method has been well developed for the case of self-adjoint boundary conditions in a spatial variable. However, for the case of non-self-conjugate boundary conditions the problem still remains open.

In this talk considers initial boundary value problems for the one-dimensional wave equation

$$u_{tt}(x,t) - u_{xx}(x,t) + q(x)u(x,t) = f(x,t),\ (x,t) \in \Omega, \tag{1}$$

in the domain $\Omega = \{(x,t) : 0 < x < 1, 0 < t < T\}$, with nonlocal boundary conditions of general form

$$U_j(u) = a_{j1}u_x(0,t) + a_{j2}u_x(1,t) + a_{j3}u(0,t) + a_{j4}u(1,t) = 0, \quad j = 1,2. \tag{2}$$

Additionally, standard initial conditions are specified

$$u(x,0) = \tau(x), \quad u_t(x,0) = \nu(x), \quad 0 \le x \le 1. \tag{3}$$

Application of the Fourier method (method of separation of variables) leads to the following spectral problem

$$-y''(x) + q(x)y(x) = \lambda y(x),\ 0 < x < 1, \tag{4}$$

$$U_j(y) = a_{j1}y'(0) + a_{j2}y'(1) + a_{j3}y(0) + a_{j4}y(1) = 0, \quad j = 1,2. \tag{5}$$

It is well known that if conditions (5) are strongly regular, then the system of root vectors of problem (4)-(5) forms a Riesz basis in $L_2(0,1)$. And the Fourier method can be implemented to solve problem (1)-(3). However, when the boundary conditions (5) are not strongly regular, the system of root vectors of problem (4)-(5) may not form an unconditional basis. And this does not make it possible to use the Fourier method. In the case when the boundary conditions (5) are irregular, the system of root vectors of problem (4)-(5) does not form an unconditional basis. Thus, for the application of the Fourier method, the case when the boundary conditions (5) are not strongly regular remains unfounded.

This talk considers just such a case. An algorithm has been constructed to prove the correctness (in the classical and generalized senses) of the initial boundary value problem (1)-(3) for the case when the boundary conditions (5) are not strongly regular. This method can be applied regardless of whether the system of root vectors of problem (4)-(5) forms an unconditional basis in $L_2(0,1)$ or not.

The technique is based on the method for solving heat conduction problems with not strongly regular boundary conditions [4].

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Keywords: wave equation, nonlocal boundary condition, not strongly regular boundary conditions, Riesz basis, Fourier method.

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Improved Hardy-Sobolev Inequalities

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In this talk, we obtain a new version of the Hardy-Sobolev inequality in the integral form which covers the recent inequality derived in [1] and improves the results established in [2]. It gives new results in one dimension. Moreover, we analyse radial and non-radial versions of the considered multidimensional sharp Hardy-Sobolev inequality and as a consequence we establish an improved version of the Heisenberg-Pauli-Weyl uncertainty principle.

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Keywords: Hardy-Sobolev inequality, Sharp Constant, Symmetric rearrangements, Uncertainty principle.

2010 Mathematics Subject Classification: 26D10, 35A23, 46E35

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On the Third-Order Differential Equation with Unbounded Coefficients

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We study the solvability and maximal regularity questions for the singular third-order differential equation with unbounded coefficients and some applications. In contrast to the cases studied earlier, the leading and intermediate coefficients of above equation can grow independently.

We obtain sufficient coefficient conditions for the correctness of the equation and compactness of the inverse to the corresponding third-order differential operator. The maximal regularity estimate for a generalized solution is proved. Using these results we have obtained upper and lower estimates for the number of Kolmogorov k-diameters of one set associated with solutions of the linear Korteweg-de Vries equation. For specific coefficients, we have shown that, two-sided estimates of the Kolmogorov k-diameters themselves are derived from the maximal regularity inequality.

Methods of the theory of closed operators, embedding and compactness theorems for weighted Sobolev spaces, as well as some integral inequalities of Hardy type with weights are used in the work.

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Keywords: singular differential equation, generalized solution, correctness, coercive estimate, resolvent, compactness.

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On Logarithmically Submajorization for τ- Measurable Operators

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We denote by $\mathcal{M}$ a semi-finite von Neumann algebra with a faithful normal finite trace τ and by $L_0(\mathcal{M})$ the set of all τ-measurable operators associated with $(\mathcal{M}, \tau)$. For $x \in L_0(\mathcal{M})$, the distribution function $\lambda.(x)$ of x is defined by $\lambda_t(x) = \tau(e_{(t,\infty)}(|x|))$ for $t > 0$, where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ in the interval (t, ∞), and the generalized singular numbers $\mu.(x)$ of x by

$$\mu_t(x) = \inf\{s > 0 : \ \lambda_s(x) \leq t\} \quad \text{for} \quad t > 0.$$

Let

$$L_{\log_+}(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \ \log_+ |x| \in L_1(\mathcal{M}) + \mathcal{M}\},$$

where $\log_+ t = \{\log t, 0\}$, $t > 0$. If $x, y \in L_{\log_+}(\mathcal{M})$ such that

$$\int_0^t \log \mu_s(x) ds \leq \int_0^t \log \mu_s(y) ds, \qquad t > 0,$$

x is said to be logarithmically submajorized by y, denoted by $x \preccurlyeq_{\log} y$.

Theorem 1. *The following statements are equivalent:*
(i) If $x, y \in L_{\log_+}(\mathcal{M})$ are self-adjoint operators such that $\pm y \leq x$, then $y \preccurlyeq_{\log} x$.
(ii) If $a, b \in \mathcal{M}$, $x, y \in L_{\log_+}(\mathcal{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$, then

$$a^* z b + b^* z^* a \preccurlyeq_{\log} a^* x a + b^* y b.$$

(iii) If $x, y, z \in L_{\log_+}(\mathcal{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$, then $z^* + z \preccurlyeq_{\log} x + y$.
(iv) If $x, y \in L_{\log_+}(\mathcal{M})$ are positive operators, then $x - y \preccurlyeq_{\log} x + y$.
(v) If $x, y, z \in L_{\log_+}(\mathcal{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geq 0$, then $z^* \oplus z \preccurlyeq_{\log} x \oplus y$.
(vi) If $x, y \in L_{\log_+}(\mathcal{M})$ are normal operators and $z \in L_{\log_+}(\mathcal{M})$ is positive operator, then for any contraction $a \in \mathcal{M}$,

$$|za(x + y)a^* z| \preccurlyeq_{\log} za(|x| + |y|)a^* z.$$

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Keywords: generalized singular value, τ-measurable operator, semi-finite von Neumann algebra.

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Finite-Time/Fixed-Time Synchronization of Memristive Shunting Inhibitory Cellular Neural Networks via Sliding Mode Control

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This abstract proposes a novel time-dependent gain parameter-based sliding mode controller (SMC) to realize the finite-time/fixed-time synchronization of memristive shunting inhibitory memristive neural networks (Mem-SICNNs) having time-varying delays. In this regard, a new terminal sliding mode surface is designed and its reachability is analysed. According to synchronization error analysis, the stability property of the desired error system is reached within finite-time/fixed-time range by proposing a unique time-dependent gain parameter-based SMC and choosing the appropriate Lyapunov functionals. Finally, a numerical example is approached by software simulation and manual calculation to estimate the settling-time of the finite-time/fixed-time synchronization criteria of the proposed Mem-SICNNs model.

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Keywords: finite-time/fixed-time synchronization; memristor; shunting inhibitory cellular neural networks; sliding mode control.

Generalized Stankovich Transform and Distributed Order Evolutionary Equations

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The report discusses an integral transform connecting the operators $\frac{d}{dx}$ and $\mathcal{D}_x^{[\mu]}$, where

$$\mathcal{D}_x^{[\mu]} f(x) = \int D_x^t f(x)\, \mu\,(dt)$$

is a distributed order differentiation operator. Here D_x^t denotes a fractional derivative of order t with respect to x, μ is an non-negative Lebesgue-Stieltjes measure. It is assumed that $\operatorname{supp}\mu \in [0,1)$ and $\sup\operatorname{supp}\mu > 0$. In the case when μ is concentrated at a point, the operator $\mathcal{D}_x^{[\mu]}$ coincides (up to a constant factor) with the fractional differentiation operator, and the corresponding transform turns into the Stankovich transform [1, 2].

As an application of the considered transform, we construct solutions of initial problems for the distributed order evolutionary equation of the form

$$\mathcal{D}_x^{[\mu]} u(x) = Lu(x) + f(x), \qquad \lim_{x\to 0} \mathcal{D}_x^{[\mu_1]} u(x) = a,$$

in terms of solutions of the problem

$$v'(x) = Lv(x) + g(x), \qquad v(x) = a.$$

Here L is a linear operator that does not depend on x (it is assumed that $u(x)$, $f(x)$ and a, as well as L, can depend on other variables, i.e. can be elements of some function space, for each fixed x), and μ_1 is the shift of the measure μ by 1.

Keywords: fractional derivative, distributed order differentiation operator, evolutionary equation, Stankovich transform.

2020 Mathematics Subject Classification: 26A33, 35A22, 35R11, 34A08

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Solvability of Boundary Value Problems for a Involutary Second-Order Differential Equation

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We are devoted to studying the existence of a solution to the differential equation

$$y''(x) + \alpha y''(-x) = F(x, y(x), y(-x)), \; x \in (-1, 1), \tag{1}$$

with boundary conditions

$$y(-1) = y_1, \, y(1) = y_2, \tag{2}$$

where $F : [0,1] \times R^2 \to R$ is a given function, $\alpha \neq \pm 1$. The boundary value problem (1), (2) is equivalent to the integral equation

$$y(x) = \frac{1}{2}(y_2 + y_1) + \frac{1}{2}(y_2 - y_1)x + \int_{-1}^{1} G(x,t) F(t, y(t), y(-t))\, dt,$$

where $G(x,t)$ is Green's function of the homogeneous boundary value problem (1), (2) in the case $y_1 = y_2 = 0$.

Theorem 1. *Let $\alpha \neq \pm 1$. Let the function $F(x, \varsigma, \xi)$ be continuous and satisfy the Lipschitz condition $\left|F(x,\varsigma,\xi) - F\left(x,\tilde{\varsigma},\tilde{\xi}\right)\right| \leq l_1 |\varsigma - \tilde{\varsigma}| + l_2 \left|\xi - \tilde{\xi}\right|$ for any $(x,\varsigma,\xi), \left(x,\tilde{\varsigma},\tilde{\xi}\right) \in \Omega$, with some positive numbers l_1, l_2 such that $\frac{9(l_1+l_2)}{16|1-|\alpha||} < 1$. Then the boundary value problem (1), (2) has a unique solution.*

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Keywords: second-order differential equation with involution; Green's function; nonlinear equation; boundary value problem; Schauder fixed point theorem.

2020 Mathematics Subject Classification: 34A34, 34B27, 34K10

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Optimal Difference Formulas in the Sobolev Spaces

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In the world today, the finite difference method is used for the numerical solution of many problems in physics and technology described by the equations of mathematical physics. The basic concepts of difference methods are approximation, stability, convergence, which are illustrated with examples of difference schemes for ordinary differential equations. For large numerical calculations, it becomes useful to optimize the process of approximate solution of ordinary differential equations, and if calculations are performed using difference formulas, then the formulas themselves are optimized. The problem of optimizing difference formulas in the modern understanding looks like the problem of finding the minimum norm of the error functional of the difference formula. Therefore, constructing optimal difference formulas in Sobolev spaces and finding the norm of the error functional of optimal difference formulas are urgent problems in computational mathematics.

This work is devoted to the study of optimal difference formulas in Sobolev space. There are algebraic and functional formulations of the problem of constructing difference formulas. In this work we consider functional formulations of the problem.

The functional formulation considers functions $\varphi(x)$, belonging to the Sobolev space $L_2^{(m)}(0,1)$, where $L_2^{(m)}$ is a Hilbert space whose elements are classes of real-valued functions differing on a polynomial of degree $(m-1)$ and square integrable with a derivative of order m on the interval $[0,1]$.

We consider approximate solution of the Cauchy problem for first order ordinary linear differential equations. For this we suggest the functional method of construction of difference formulas. The error of the difference formula is estimated from above by the norm of the error functional of this formula. To find in explicit form the norm of the error functional ℓ of difference formulas, we use the extremal function of this functional. Next, we find the extremal function in the Sobolev space $L_2^{(m)}(0,1)$ for any $m \geq 2$.

By minimizing the squared norm of the error functional ℓ by the coefficients $C^{(1)}[\beta]$ of difference formulas, we obtain a Wiener–Hopf type system for finding optimal coefficients $\mathring{C}^{(1)}[\beta]$ difference formulas and optimal polynomial of discrete argument $\mathring{P}_{m-2}[\beta]$. Here we prove the existence and uniqueness of a solution to a Wiener–Hopf type system. At the same time, this work developed an algorithm for constructing optimal difference formulas in the Sobolev space $L_2^{(m)}(0,1)$ for any $m \geq 2$, with the help of which representations of the coefficients of optimal Adams-type difference formulas are obtained.

Keywords: optimal difference formulas, the error functional, Sobolev space, ordinary differential equations, approximation, stability, convergence.

2020 Mathematics Subject Classification: 65L05, 34A12

Inverse Problem for Fractional Order Subdiffusion Equation

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We study the inverse problem of finding functions $\{u(t), f\}$ that satisfy the following subdiffusion equation

$$D_t^\rho u(t) + Au(t) = g(t)f, \quad \rho \in (0,1], \quad t \in (0,T], \tag{1}$$

with the initial

$$u(0) = \varphi, \tag{2}$$

and the additional conditions

$$\int_0^T u(t)dt = \psi. \tag{3}$$

Here $g(t) \in C[0,T]$ is a given function and $\varphi, \psi \in H$ are known elements, A is an unbounded positive self-adjoint operator and D_t^ρ stands for the Caputo fractional derivative.

We introduce the power of operator A with domain

$$D(A) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^2 |h_k|^2 < \infty\},$$

acting in H according to the rule:

$$Ah = \sum_{k=1}^{\infty} \lambda_k h_k v_k.$$

For $0 < \rho < 1$ and an arbitrary complex number μ, let $E_{\rho,\mu}(z)$ denote the Mittag-Leffler function with two parameters of the complex argument z (see [1]):

$$E_{\rho,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + \mu)}.$$

Lemma 2. *Let $\rho \in (0,1]$, $g(t) \in C^1[0,T]$ and $g(0) \neq 0$. Then there exist numbers $m_0 > 0$ and k_0 such that, for all $T \leq m_0$ and $k \geq k_0$, the following estimates hold:*

$$\frac{C_0}{\lambda_k} \leq |p_{k,\rho}(T)| \leq \frac{C_1}{\lambda_k},$$

where

$$p_{k,\rho}(T) = \int_0^T g(\eta)(T-\eta)^\rho E_{\rho,\rho+1}(-\lambda_k (T-\eta)^\rho) d\eta$$

and constants C_0 and $C_1 > 0$ depend on m_0 and k_0.

Let $\mathbb{N} = K_\rho \cup K_{0,\rho}$, where $\mathbb{N}$ is the set of all natural numbers. K_ρ and $K_{0,\rho}$ are sets such that: if $p_{k,\rho}(T) \neq 0$, $k \in K_\rho$, otherwise, if $p_{k,\rho}(T) = 0$, then $k \in K_{0,\rho}$.

Theorem 1. *Let $\rho \in (0,1]$, $\varphi \in H$, $\psi \in D(A)$, $g(t) \in C[0,T]$ and $g(t) \neq 0$, $t \in [0,T]$. Then there exists a unique solution of the inverse problem (1)-(3).*

Theorem 2. *Let $\rho \in (0,1]$, $\varphi \in H$, $\psi \in D(A)$, $g(t) \in C^1[0,T]$. Further, we will assume that the conditions of Lemma 2 are satisfied and T is sufficiently small. If set $K_{0,\rho}$ is empty, for all k, then there exists a unique solution of the inverse problem (1)-(3):*

$$f = \sum_{k=1}^{\infty} \frac{1}{p_{k,\rho}(T)} \left[\psi_k - \varphi_k T E_{\rho,2}(-\lambda_k T^\rho)\right] v_k,$$

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k E_{\rho,1}(-\lambda_k t^\rho) + \frac{p_{k,\rho}(t)}{p_{k,\rho}(T)} \left[\psi_k - \varphi_k T E_{\rho,2}(-\lambda_k T^\rho)\right] \right] v_k.$$

If set $K_{0,\rho}$ is not empty, then for the existence of a solution to the inverse problem, it is necessary and sufficient that the following conditions

$$\psi_k = \varphi_k T E_{\rho,2}(-\lambda_k T^\rho), \quad k \in K_{0,\rho}$$

be satisfied. In this case, the solution to the problem (1)-(3) exists, but is not unique:

$$f = \sum_{k\in B_\rho} \frac{1}{p_k(T)} \left[\psi_k - \varphi_k T E_{\rho,2}(-\lambda_k T^\rho)\right] v_k + \sum_{k\in K_{0,\rho}} f_k v_k,$$

$$u(t) = \sum_{k=1}^{\infty} \left[\varphi_k E_{\rho,1}(-\lambda_k t^\rho) + f_k\right] v_k,$$

where f_k, $k \in K_{0,\rho}$, are arbitrary real numbers.

Keywords: subdiffusion equation, inverse problem, the Caputo derivative, Fourier method.

2020 Mathematics Subject Classification: Primary 35R11; Secondary 34A12.

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Quality properties of multipoint problem for Schrodinger equations and applications

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The existence, uniqueness, regularity properties and Strichartz type estimates for the solution of multipoint Cauchy problem for linear and nonlinear Schrödinger equations with general elliptic leading part are obtained.

Key Word: Schrődinger equations**,** elliptic operators**,** local solutions, Strichartz type inequalities, regularity properties of PDE

Consider the multipoint Cauchy problem for nonlinear Schrödinger equations (NLS)

$$i\partial_t u + Lu + F(u) = 0,\ x \in \mathbb{R}^n,\ t \in [0,T], \tag{1.1}$$

$$u(0,x) = \varphi(x) + \sum_{k=1}^{m} \alpha_k u(\lambda_k, x),\ \text{for a.e. } x \in \mathbb{R}^n, \tag{1.2}$$

where L is an elliptic operator defined by

$$Lu = \sum_{|\beta| \leq 2l} a_\beta D^\beta u, \tag{1.3}$$

for $\beta = (\beta_1, \beta_2, ..., \beta_n)$, $a_\beta \in \mathbb{C}$, m is an integer, $\lambda_k \in (0,\ T]$, α_k are complex numbers, F is a nonlinear operator and $u = u(t,x)$ is an unknown function. If $F(u) = \lambda |u|^p u$ in (1.1) we get the multipoint Cauchy problem for the nonlinear equation

$$i\partial_t u + Lu + \lambda |u|^p u = 0,\ x \in \mathbb{R}^n,\ t \in [0,T], \tag{1.4}$$

$$u(0,x) = \varphi(x) + \sum_{k=1}^{m} \alpha_k u(\lambda_k, x)\ \text{ for a.e. } x \in \mathbb{R}^n,$$

where $p \in (1,\infty)$, λ is a real number.

The existence of solutions and regularity properties of Cauchy problem for NLS equations studied e.g in $[1-5]$, [7], [8] and the references therein. In contrast, to the mentioned above results we will study the existence, uniqueness and the regularity properties of the multipoint Cauchy problem $(1.1)-(1.2)$.

2. Definitions and background

Let $L_t^q L_x^r((a,b)\times\Omega)$ denotes the space of strongly measurable functions that are defined on the measurable set $(a,b)\times\Omega$ with the norm

$$\|f\|_{L_t^q L_x^r((a,b)\times\Omega)} = \left({}_a^b \left(\int_\Omega |f(t,x)|^r \, dx \right)^{\frac{q}{r}} dt \right)^{\frac{1}{q}},\ 1 \leq q, r < \infty\ .$$

Let $\mathcal{F}$ denotes the Fourier transformation, $\hat{u} = \mathcal{F}u$ and

$$s \in \mathbb{R},\ \xi = (\xi_1, \xi_2, ..., \xi_n) \in \mathbb{R}^n,\ |\xi|^2 = {}_{k=1}^{n} \xi_k^2.$$

$S = S(\mathbb{R}^n)$ denotes the Schwartz class, i.e. the space of all complex-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by semi-norms. $S(\mathbb{R}^n)$ denoted by just S. Let $S'(\mathbb{R}^n)$ denote the space of all continuous linear operators, $L : S \to \mathbb{C}$, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n)$ is norm dense in $L^p(\mathbb{R}^n)$ when $1 < p < \infty$. Let $D'(\Omega)$ denote the class of generalized functions on $\Omega \subset \mathbb{R}^n$. Consider Sobolev space $W^{s,p}(\mathbb{R}^n)$ and homogeneous Sobolev spaces $\mathring{W}^{s,p}(\mathbb{R}^n)$ defined by respectively,

$$W^{s,p}(\mathbb{R}^n) = \Big\{u : u \in S'(\mathbb{R}^n),$$

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} = \left\| F^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{u}\right\|_{L^p(\mathbb{R}^n)} < \infty \Big\},$$

$$\mathring{W}^{s,p}(\mathbb{R}^n) = \left\{u : u \in S'(R^n), \|u\|_{\mathring{W}^{s,p}(\mathbb{R}^n)} = \left\|F^{-1}|\xi|^s \hat{u}\right\|_{L^p(\mathbb{R}^n)} < \infty\right\}.$$

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say η, we write C_η.

Let L be a differential operator defined by (1.3) and $A = [a_{ij}]$, $i, j = 1, 2, ..., n$.

Condition 2.1. Let $L_0(\xi) =_{|\beta|=2l} a_\beta \xi^\beta \neq 0$ for $\xi \in \mathbb{R}^n$, $\xi \neq 0$ and there exists a positive constant M_0 such that

$$|L(\xi)| = \left|_{|\beta|\leq 2l} a_\beta (i\xi)^\beta\right| \geq M_0\left(|\xi|^{2l}+1\right)$$

for $\xi = (\xi_1, \xi_2, ...\xi_n) \in \mathbb{R}^n$ with $\xi^\beta = \xi_1^{\beta_1}.\xi_2^{\beta_2}...\xi_n^{\beta_n}$.

Definition 2.2. Consider the initial value problem $(1.1)-(1.2)$ for $\varphi \in \mathring{W}^{s,p}(\mathbb{R}^n)$. This problem is critical when $s = s_c := \frac{n}{2} - \frac{2}{p}$, subcritical when $s > s_c$, and supercritical when $s < s_c$.

We write $a \lesssim b$ to indicate that $a \leq Cb$ for some constant C, which is permitted to depend on some parameters.

3. Dispersive and Strichartz type inequalities for linear Schrödinger equation

Assume the Condition 2.1 holds. The fundamental solution of the Schrödinger operator $i\partial_t u + Lu$ is found as solution of the equation

$$i\partial_t u + Lu = \delta(t,x), \tag{3.1}$$

where $\delta(t,x)$ is the generalized delta function. By applying the Fourier transformation with respect to x in (3.1) we get

$$i\frac{d}{dt}\hat{u}(t,\xi) - L(\xi)\hat{u}(t,\xi) = 1(\xi) \times \delta(t). \tag{3.2}$$

By using the Fourier transformation and by following [9, § 6.6,14.4] it can be shown that $\hat{u}(t,\xi) = \theta(t) e^{-itL(\xi)}$, i.e. the fundamental solution (3.1) can be exspressed as

$$U(t,.) = U_L(t,x) = F_x^{-1}\left[\theta(t) e^{-itL(\xi)}\right],$$

where F_x is the Fourer transformation with respect to x and $\theta(t)$ is the Heaviside unit function.

Condition 3.1. Assume $n \geq 1$,

$$\frac{2}{q} + \frac{n}{r} \leq \frac{n}{2},\ 2 \leq q, r \leq \infty \text{ and } (n,\ q,\ r) \neq (2,\ 2,\ \infty). \tag{3.3}$$

Assume H is an abstract Hilbert space and Q is a Hilbert space of function. Suppose for each $t \in \mathbb{R}$ an operator $U(t)$: $Q \to L^2(\Omega)$ obeys the following estimates:

$$\|U(t) f\|_{L_x^2(\Omega)} \lesssim \|f\|_H \tag{3.4}$$

for all t, $\Omega \subset \mathbb{R}^n$ and all $f \in Q$;

$$\|U(s) U^*(t) g\|_{L_x^\infty(\Omega)} \lesssim |t-s|^{-\frac{n}{2}} \|g\|_{L_x^1(\Omega)}, \tag{3.5}$$

$$\|U(s) U^*(t) g\|_{L_x^\infty(\Omega)} \lesssim \left(1 + |t-s|^{-\frac{n}{2}}\right) \|g\|_{L_x^1(\Omega)} \tag{3.6}$$

for all $t \neq s$ and all $g \in L_x^1(\Omega)$.

Consider the multipoint Cauchy problem for forced Schrodinger equation

$$i\partial_t u + Lu = f,\ t \in [0,T],\ x \in \mathbb{R}^n, \tag{3.7}$$

$$u(t_0, x) = \varphi(x) + \sum_{k=1}^m \alpha_k u(\lambda_k, x),\ x \in \mathbb{R}^n,\ t_0,\ \lambda_k \in [0,\ T\ ),\ \lambda_k > t_0.$$

We are now ready to state the standard Strichartz estimates:

Lemma 3.3. Assume the Condition 2.1 is satisfied, $\varphi \in \mathring{W}^{\gamma,p}(\mathbb{R}^n)$ for $\gamma \geq \frac{n}{p}$ and $p \in [1,\infty]$. Then problem (3.7) has a unique generalized solution.

Proof. By using the Fourier transform we get from (3.7) :

$$i\hat{u}_t(t,\xi) - L(\xi)\hat{u}(t,\xi) = \hat{f}(t,\xi), \tag{3.8}$$

$$\hat{u}(0,\xi) = \hat{\varphi}(\xi) + \sum_{k=1}^m \alpha_k \hat{u}(\lambda_k, \xi),\ \text{for a.e. } \xi \in \mathbb{R}^n.$$

where $\hat{u}(t,\xi)$ is a Fourier transform of $u(t,x)$ with respect to x.

Consider the problem

$$\hat{u}_t(t,\xi) + iL(\xi)\hat{u}(t,\xi) = \hat{f}(t,\xi), \tag{3.9}$$

$$\hat{u}(0,\xi) = u_0(\xi),\ \xi \in \mathbb{R}^n,\ t \in [0,T],$$

where $u_0(\xi) \in \mathbb{C}$ for $\xi \in \mathbb{R}^n$. In view of Condition 2.1, by [6, §11] $-iL(\xi)$ is a generator of a strongly continuous C_0 groups $U_L(t,\xi) = e^{-itL(\xi)}$. Hence, the Cauchy problem (3.9) has a solution for all $\xi \in \mathbb{R}^n$ and the solution can be expressed as

$$\hat{u}(t,\xi) = e^{-itL(\xi)} u_0(\xi) + \int_{t_0}^{t} e^{-itL(\xi)|t-\tau|} \hat{f}(\tau,\xi)\, d\tau,\ t \in (0,T). \tag{3.10}$$

Using the formulas $(3.8) - (3.10)$, we get that the solution of (3.7) will be expressed as the following formula:

$$u(t,x) = V(t)\varphi(x) + \sum_{k=1}^m \alpha_k V_k(t,x) + \sum_{k=1}^m \alpha_k G_k(t,x) + G_0(t,x),$$

where

$$V(t) = F^{-1}\left[U_L(t,\xi)\hat{\varphi}(\xi)\right],\ V_k(t,x) = F^{-1}\left[U_L(\lambda_k,\xi)\hat{\varphi}(\xi)\right], \tag{3.11}$$

$$G_k(t,x) = F^{-1}\left[\int_{t_0}^{\lambda_k} U_L(\lambda_k - \tau, \xi)\hat{f}(\tau,\xi)\, d\tau\right],$$

$$G_0(t,x) = F^{-1}\left[\int_{t_0}^{t} U_L(t-\tau,\xi)\hat{f}(\tau,\xi)\, d\tau\right].$$

By following [7, Theorem 1.2] we have:

Theorem 3.1. Assume $U(t)$ obeys (3.5) and (3.6). Then the following estimates are hold

(3.12) $$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_H,$$

(3.13) $$\|U^*(s)f(s)\,ds\|_Q \lesssim \|f\|_{L_t^{q'} L_x^{r'}},$$

(3.14) $$_{s<t}\|U(t)U^*(s)f(s)\,ds\|_{L_t^q L_x^r} \lesssim \|f\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}$$

for all pairs (q, r), $(\tilde{q}, \tilde{r})$.

Proof. By duality, (3.13) is equivalent to (3.14). By the TT^* method, (3.14) is in turn equivalent to the bilinear form estimate

(3.15) $$|\langle U^*(s)f(s), U^*(t)G(t)\rangle dsdt| \lesssim \|f\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}}.$$

By symmetry it suffices to show the retarded version of (3.13)

(3.16) $$|T(f, G)| \lesssim \|f\|_{L_t^{q'} L_x^{r'}} \|G\|_{L_t^{q'} L_x^{r'}},$$

where $T(F, G)$ is the bilinear form defined by

$$T(F, G) = \int\int_{s<t} \langle U(s)^* f(s), (U(t))^* G(t)\rangle dsdt.$$

Then by reasoning as in [10, Theorem 3.1], by using the estimates (3.15), (3.16) and the Lemma 3.3, we get the conclusion.

Theorem 3.2. Assume the Conditions 2.1 and 3.1 are satisfied. Let $0 \leq s \leq 1$, $\varphi \in \mathring{W}^{s,2}(\mathbb{R}^n)$, $f \in N^0\left([0, T]; \mathring{W}^{s,2}(\mathbb{R}^n)\right)$ and let $u : \sigma \to \mathbb{C}$ be a solution to (3.7). Then

(3.17) $$\||\nabla|^s u\|_{S^0([0,T])} + \||\nabla|^s u\|_{C^0([0,T]; L^2(\mathbb{R}^n))} \lesssim$$

$$\||\nabla|^s \varphi\|_{L^2(\mathbb{R}^n)} + \||\nabla|^s f\|_{N^0([0,T])}.$$

Proof. Let $2 \leq q, r, \tilde{q}, \tilde{r} \leq \infty$ with

$$\frac{2}{q} + \frac{n}{r} = \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2}.$$

If $n = 2$, we also require that q, $\tilde{q} > 2$. Consider first, the nonendpoint case. By Lemma 3.3 the problem has a solution. The linear operators in (3.12)-(3.14) are adjoint of one another; thus, by the method of TT^* both will follow once we prove

$$\left\|\int_{s<t} U_L(t-s,.)f(s)\,ds\right\|_{L_t^q L_x^r} \lesssim \|f\|_{L_t^{q'} L_x^{r'}}.$$

Apply Theorem 3.1 with $Q = L_x^2(\mathbb{R}^n) = L_x^2$. The energy estiamate (3.12):

$$\|U_L(t,.)f\|_{L_x^2} \lesssim \|f\|_{L_x^2}$$

follows from Plancherel's theorem, the untruncated decay estimate

$$\|U_L(t-s,.)f\|_{L_x^\infty} \lesssim |t-s|^{-\frac{n}{2}} \|f\|_{L_x^1},$$

and explicit representation of the Schredinger evolution operator $U_L(t) = U_L(t, x)$. Due to properties gropes $U_L(t)$ and by the dispersive estimate (3.4) we have

$$|\Phi| \lesssim_{s<t} |U_L(t-s)\,ds|_{B(H)} |f(s)|\,ds \lesssim_{\mathbb{R}} |t-s|^{-n\left(\frac{1}{2}-\frac{1}{p}\right)} |f(s)|\,ds,$$

where

$$\Phi = \int_{s<t} U_L(t-s)f(s)\,ds.$$

Then by reasoning as in [10, Theorem 3.2], we obtain the conclusion.

4. Strichartz type estimates for solution of the nonlinear Schrödinger equation

Consider the multipoint initial-value problem $(1.1)-(1.2)$.

Condition 4.1. Assume that the function $F:\mathbb{C}\to\mathbb{C}$ is continuously differentiable and obeys the power type estimates

$$F(u)=O\left(|u|^{1+p}\right),\ F_u(u)=O\left(|u|^p\right), \tag{4.1}$$

$$F_u(\upsilon)-F_u(w)=O\left(|\upsilon-w|^{\min\{p,1\}}+|w|^{\max\{0,p-1\}}\right) \tag{4.2}$$

for some $p>0$, where $F_u(u)$ denotes the derivative of operator function F with respect to u.

From (4.1) we obtain

$$|F(u)-F(\upsilon)|\lesssim|u-\upsilon|\left(|u|^p+|\upsilon|^p\right). \tag{4.3}$$

Remark 4.1. The model example of a nonlinearity obeying the conditions above is $F(u)=|u|^p\,u$, $p\in(1,\infty)$ for which the critical homogeneous Sobolev space is $\mathring{W}_x^{s_c,2}(\mathbb{R}^n)$ with $s_c:=\frac{n}{2}-\frac{2}{p}$.

Definition 4.1. A function $F:\sigma\to\mathbb{C}$ is called a (strong) solution to $(1.1)-(1.2)$ if it lies in the class

$$C_t^0\left([0,T];\mathring{W}_x^{s,2}(\mathbb{R}^n)\right)\cap L_t^{p+2}L_x^{\frac{np(p+2)}{4}}(\sigma)$$

and obey:

$$u(t,x)=V(t)\varphi(x)+\sum_{k=1}^{m}\alpha_kV_k(t,x)+\sum_{k=1}^{m}\alpha_kG_k(t,x)+G_0(t,x), \tag{4.4}$$

where

$$V(t)=F^{-1}\left[U_L(t,\xi)\hat{\varphi}(\xi)\right],\ V_k(t,x)=F^{-1}\left[U_L(\lambda_k,\xi)\hat{\varphi}(\xi)\right],$$

$$G_k(t,x)=F^{-1}\left[\int_{t_0}^{\lambda_k}U_L(\lambda_k-\tau,\xi)\hat{F}(\tau,\xi)\,d\tau\right], \tag{4.5}$$

$$G_0(t,x)=F^{-1}\left[\int_{t_0}^{t}U_L(t-\tau,\xi)\hat{F}(\tau,\xi)\,d\tau\right].$$

We say that u is a global solution if $T=\infty$.

Theorem 4.1. Assume the Condıtons 2.1, 3.1, 4.1 are satisfied. Let $0\le s\le 1$, $\varphi\in\mathring{W}^{s,2}(\mathbb{R}^n)$ and $n\ge 1$. Then there exists $\eta_0=\eta_0(n)>0$ such that if $0<\eta\le\eta_0$ such that

$$\left\|\left|\nabla\right|^s U_L(t)\varphi\right\|_{L_t^{p+2}L_x^{\sigma}(\sigma)}\le\eta,$$

then here exists a unique solution u to (4.1) on $[0,T]\times\mathbb{R}^n$. Moreover, the following estimates hold

$$\left\|\left|\nabla\right|^s U_Lu\right\|_{L_t^{p+2}L_x^{\sigma}(\sigma)}\le 2\eta,$$

$$\left\|\left|\nabla\right|^s u\right\|_{S^0(\sigma)}+\|u\|_{C^0\left([0,T];\mathring{W}^{s,2}(\mathbb{R}^n)\right)}\lesssim\left\|\left|\nabla\right|^s\varphi\right\|_{L_x^2(\mathbb{R}^n)}+\eta^{1+p},$$

$$\|u\|_{S^0(\sigma;H)}\lesssim\|\varphi\|_{L_x^2(R^n;H)},\ r=r(p,n)=\frac{2n(p+2)}{2(n-2)+np}.$$

Proof. We apply the standard fixed point argument. More precisely, using the estimates (3.26), the equalitie (3.11) and by reasoning as in [10, Theorem 4.1], we obtain the conclusion.

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Time Nonlocal Schrödinger Equation with Spatial Periodic Boundary Condition in Fractional Power Hilbert Spaces

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In this study, time nonlocal Schrödinger equation with spatial periodic boundary condition in fractional power Hilbert spaces is investigated. Stable first order of accuracy Rothe difference scheme and second order of accuracy Crack-Nicholson difference scheme for the numerical solution of this problem is presented. The main theorem on the stability of these difference schemes are established. Numerical results are given.

Keywords: difference schemes, stability, Schrödinger equation

2020 Mathematics Subject Classification: 35J25, 47E05, 34B27

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Numerical Algorithms for Solving the Inverse Problem of Identifying the Initial Conditions of a Subdiffusion Equation

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The paper considers the parallel algorithm for solving the retrospective inverse problem of identifying the initial value for a subdiffusion equation. To solve this inverse problem, the regularized iterative conjugate gradient method is used. At each iteration of the method, we need to solve the auxiliary direct initial-value problem. Using the finite difference scheme, the direct problem is reduced to a large number of systems of linear algebraic equations.

The following subdiffusion equation is considered:

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \mathcal{L}u = f(\mathbf{x}, t), \quad \mathbf{x} \in \Omega, 0 < t \leq T,$$

$$u(\mathbf{x}, t) = 0,\ \mathbf{x} \in \partial\Omega,\ 0 < t \leq T,$$

$$u(\mathbf{x}, 0) = \varphi(\mathbf{x}), \mathbf{x} \in \Omega.$$

Here, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \overline{\Omega} = \prod_{i=1}^{n} [a_i, b_i],\ 0 < \alpha < 1$, and $\mathcal{L}$ is an elliptic operator

$$\mathcal{L}u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(k_i(\mathbf{x}, t)\frac{\partial u}{\partial x_i}\right), x_i \in (a_i, b_i), 0 < t \leq T.$$

The fractional Caputo derivative with order α is defined as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s}(t-s)^{-\alpha}, \mathbf{x} \in \Omega, 0 < t \leq T.$$

In the present work, we study the inverse problem of restoring the initial condition $\varphi(\mathbf{x})$. Additional information on the solution is given in the form

$$u(\mathbf{x}, t) = \varphi(\mathbf{x}), \mathbf{x} \in \Omega.$$

For the numerical solution of the inverse problem, the iterative conjugate gradient method [1] is used, while at each iteration the direct problem is solved by using an implicit difference scheme [2]. The direct problem is reduced to solving a large system of linear algebraic equation with triadiagonal (for the 1D case) or block tridiagonal (for the 2D and larger dimensions) matrix at each time step.

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Keywords: fractional differential equations, inverse problems, retrospective problem.

2020 Mathematics Subject Classification: 35R30, 65R32

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Maximizing Schatten p-Norms and Related Isoperimetric Inequalities

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In this presentation, we explore the intriguing realm of Schatten p-norms and their relationship with Riesz potential operators within domains of fixed measure. Our main result unveils the ball as an optimal maximizer for these integer order Schatten p-norms. This finding extends to the polyharmonic Newton potential operator, a vital component in nonlocal boundary value problems associated with the poly-Laplacian. This extension mirrors the pioneering work of M. Kac and T. Kalmenov, originally established for the Laplacian, leading us to the derivation of isoperimetric inequalities for the eigenvalues. These inequalities, akin to the classical Rayleigh–Faber–Krahn and Hong–Krahn–Szego counterparts, further illuminate the structural properties of the polyharmonic Newton potential operator.

In our presentation, we delve into the broader landscape by considering extensions of these results to convolution-type integral operators, offering explicit examples to elucidate our findings. Additionally, we present a new insight into the multidimensional MEMS (micro-electro mechanical systems) problem within the Euclidean space R^d, where $d \geq 3$. Our investigation reveals that minimizing the pull-in voltage for this problem involves the symmetrization of the permittivity profile, and we establish the foundation for this claim through the application of Talenti's comparison principle. This talk is based on our joint works [1]-[3].

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Keywords: MEMS problem, pull-in voltage, Newton potential, Talenti's comparison principle, geometric estimate.

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Critical Exponents for the Exterior Evolution Problems

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The paper studies the large-time behavior of solutions to the Robin problem for PDEs with critical nonlinearities. For the considered problems, nonexistence results are obtained, which complements the interesting recent results by Ikeda et al. [1], where critical cases were left open. Moreover, our results provide partially answers to some other open questions previously posed by Zhang [2] and Jleli-Samet [3]. The proof of main results is based on methods of nonlinear capacity estimates specifically adapted to the nature of the exterior domain. Furthermore, the difference in our approach lies in the fact that we are considering a class of test functions with logarithmic arguments.

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Keywords: exterior Robin problem, critical exponent, nonexistence, global solution.

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On Fourier Multiplier on Non-Commutative Torus and Applications

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In this work, we deal with the Fourier multiplier on non-commutative torus and its applications in non-commutative Harmonic analysis.

On Solvability of Some Inverse Problems for a Nonlocal Fourth-Order Parabolic Equation with Multiple Involution

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In this paper, the solvability of some inverse problems for a nonlocal analogue of a fourth-order parabolic equation is studied. For this purpose, a nonlocal analogue of the biharmonic operator is introduced. When defining this operator, transformations of the involution type are used. In a parallelepiped, the eigenfunctions and eigenvalues of the Dirichlet type problem for a nonlocal biharmonic operator are studied. The eigenfunctions and eigenvalues for this problem are constructed explicitly and the completeness of the system of eigenfunctions is proved. Two types of inverse problems on finding a solution to the equation and its right-hand side are studied. In the first problem, the right-hand side depending on the spatial variable is sought, and in the second problem, a function depending on the time variable is found. The first problem is solved by using the Fourier variable separation method, and the second problem by reducing it to solving an integral equation. The theorems on the existence and uniqueness of the solution are proved.

Note that similar problems in the case $n = 2$ of a classical parabolic equation in a rectangular domain were studied in [1,2].

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Keywords: inverse problem, nonlocal biharmonic operator, parabolic equation, eigenfunction, eigenvalue, Fourier method, existence of solution, uniqueness of solution.

2020 Mathematics Subject Classification: 34K06, 34K08, 35G16

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Direct and Inverse Problems for the Burgers Equation in Degenerating Domains

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Let consider the following nonlinearly degenerating domain

$$\Omega = \{x, t \,|\, \varphi_1(t) < x < \varphi_2(t),\ 0 < t < T < \infty\},$$

with its cross section $\Omega_t = \{\varphi_1(t) < x < \varphi_2(t)\}$ for a fixed value of the time variable $t \in (0, T)$, with

$$\varphi_1(0) = \varphi_2(0).$$

The functions $\varphi_1(t)$ and $\varphi_2(t)$ are defined on $[0, T]$, and belong to $C^1(0, T)$.

In the domain Ω consider the following inverse problem for the Burgers equation

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) - \nu \partial_x^2 u(x,t) = w(t) f_t(x), \text{ in } \Omega, \tag{1}$$

$$\partial_x^j u(\varphi_1(t), t) = \partial_x^j u(\varphi_2(t), t),\ j = 0,\ 1;\ t \in (0, T), \tag{2}$$

$$u(0, 0) = 0, \tag{3}$$

$$\int_{\varphi_1(t)}^{\varphi_2(t)} u(x,t)dx = E(t),\ t \in (0, T), \tag{4}$$

where $\nu = \mathit{const} > 0$ is a given constant and functions $f_t(x)$, $E(t)$ satisfy the conditions

$$\begin{cases} f_t(x) \equiv f(x,t) \in L^\infty(0, T; L^\infty(\Omega_t)),\ E(t) \in W^{1,\infty}(0, T),\ E(0) = 0, \\ \widetilde{f}(t) \equiv \int\limits_{\varphi_1(t)}^{\varphi_2(t)} f_t(x)dx \neq 0,\ \forall t \in (t_0, T),\ t_0 > 0,\ \widetilde{f}(t) = O(\varphi(t)),\ t \to 0+, \end{cases} \tag{5}$$

here $\varphi(t) = \varphi_2(t) - \varphi_1(t)$.

The work will study the question of solvability of the inverse problem, namely, what conditions must be satisfied by the functions along which the domain changes in order for the inverse problem to be uniquely solvable. This work is a logical continuation of works [1,2].

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Keywords: Inverse problem, Burgers equation, non-parabolic domain, periodic boundary condition, integral overdetermination condition.

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Hardy and Rellich Type Identities and Inequalities Related to Baouendi-Grushin Operator

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Recall from [1] that the Hardy inequality for Baouendi-Grushin vector fields takes the form

$$\int_{\mathbb{R}^n} \left(|\nabla_x f|^2 + |x|^{2\gamma} |\nabla_y f|^2\right) dz \geq \left(\frac{Q-2}{2}\right)^2 \int_{\mathbb{R}^n} \left(\frac{|x|^{2\gamma}}{|x|^{2+2\gamma} + (1+\gamma)^2 |y|^2}\right) |f|^2 dz, \tag{1}$$

where $z = (x_1, ..., x_m, y_1, ..., y_k) = (x, y) \in \mathbb{R}^m \times \mathbb{R}^k$ with $n = m + k,\ m, k \geq 1,\ \gamma \geq 0,\ Q = m + (1+\gamma)k$ and $f \in C_0^\infty\left(\mathbb{R}^m \times \mathbb{R}^k \backslash \{(0,0)\}\right)$. Here, $\nabla_x f$ and $\nabla_y f$ are the gradients of f in the variables x and y, respectively.

In this talk, we discuss sharp remainder formulae for Hardy and Rellich inequalities related to the Baouendi-Grushin operator involving radial derivatives in some of the variables, which improve the classical Hardy and Rellich inequalities as well as (1). If time permits, we also discuss magnetic functional inequalities related to the Baouendi-Grushin operator with Aharonov-Bohm type magnetic field.

This talk is based on the joint research with Ari Laptev (Imperial College London, UK) and Michael Ruzhansky (Ghent University, Belgium) [2]-[3], and with Amir Zhangirbayev.

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Keywords: Hardy inequalities, Baouendi-Grushin operator, Aharonov-Bohm magnetic field.

2010 Mathematics Subject Classification: 26D10, 35P15

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Iterated Discrete Hardy-Type Inequalities: the Case $\theta < P$

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Let $0 < q, p, \theta < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\varphi = \{\varphi_i\}_{i=1}^{\infty}$ be a sequence of non-negative numbers, $u = \{u_i\}_{i=1}^{\infty}$ and $w = \{w_i\}_{i=1}^{\infty}$ be sequences of positive numbers, which will be called the weight sequences. Let us denote by $l_{p,u}$ the space of all sequences $f = \{f_i\}_{i=1}^{\infty}$ of real numbers such that

$$\|f\|_{p,u} = \left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty.$$

For any $f \in l_{p,u}$ we characterize the following iterated discrete Hardy-type inequality with three weights

$$\left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=1}^{n} \left|\varphi_k \sum_{i=1}^{k} f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \leq C \left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}}, \tag{1}$$

where C is a positive constant independent of f. The dual discrete version of inequality (1) has the form

$$\left(\sum_{n=1}^{\infty} w_n^{\theta} \left(\sum_{k=n}^{\infty} \left|\varphi_k \sum_{i=k}^{\infty} f_i\right|^q\right)^{\frac{\theta}{q}}\right)^{\frac{1}{\theta}} \leq C \left(\sum_{i=1}^{\infty} |u_i f_i|^p\right)^{\frac{1}{p}}. \tag{2}$$

Paper [1] also contains results for inequality (2) for the case $0 < p \leq 1$, but when $p \leq \min\{q, \theta\} < \infty$. In paper [2], discrete Hardy-type inequality (1) have been characterized for the same relations between p, θ and q, namely, for the cases $p \leq \theta < \infty$, $0 < q$ and $\theta < p < \infty$, $0 < q < \theta$. Here we consider the most difficult case $\theta < p < \infty$ and $0 < \theta < q$ or, equivalently, $0 < \theta < \min\{p, q\} < \infty$, which has no explicit

Theorem 1. *Let $0 < \theta < \min\{p, q\} < \infty$, $p > 1$. Then inequality (1) holds if and only if $B_2 < \infty$, where*

$$B_2 = \left[\sum_{i=1}^{\infty} u_i^{-p'} \left(\sum_{j=1}^{i} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=i}^{\infty} w_n^{\theta} \left(\sum_{k=i}^{n} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_2$, where C is the best constant in (1).

Theorem 2. *Let $0 < \theta < \min\{p, q\} < \infty$, $p > 1$. Then inequality (2) holds if and only if $B_1 < \infty$, where*

$$B_1 = \left[\sum_{i=1}^{\infty} u_i^{-p'} \left(\sum_{j=i}^{\infty} u_j^{-p'}\right)^{\frac{p(\theta-1)}{p-\theta}} \left(\sum_{n=1}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_1$, where C is the best constant in (2).

Theorem 3. *Let $0 < \theta < \min\{p, q\} < \infty$, $0 < p \leq 1$. Then inequality (1) holds if and only if $B_4 < \infty$, where*

$$B_4 = \left[\sum_{i=1}^{\infty} u_i^{-\frac{\theta p}{p-\theta}} \left(\sum_{n=i}^{\infty} w_n^{\theta} \left(\sum_{k=i}^{n} \varphi_k^q\right)^{\frac{\theta}{q}}\right)^{\frac{p}{p-\theta}}\right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_4$, where C is the best constant in (1).

Theorem 4. *Let $0 < \theta < \min\{p, q\} < \infty$, $0 < p \leq 1$. Then inequality (2) holds if and only if $B_3 < \infty$, where*

$$B_3 = \left[\sum_{i=1}^{\infty} u_i^{-\frac{\theta p}{p-\theta}} \left(\sum_{n=1}^{i} w_n^{\theta} \left(\sum_{k=n}^{i} \varphi_k^q \right)^{\frac{\theta}{q}} \right)^{\frac{p}{p-\theta}} \right]^{\frac{p-\theta}{p\theta}}.$$

Moreover, $C \approx B_3$, where C is the best constant in (2).

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Keywords: discrete inequality, Hardy-type operator, weight sequence, sequence space, characterization.

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Construction of Stable Control System for a Given Program Manifold

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We consider the problem of construction of control automatic systems by given $(n-s)$-dimensional program manifold $\Omega(t) \equiv \omega(t,x) = 0$, in the following form:

$$\dot{x}(t) = f(t,x) - B_1\xi, \quad \xi = \varphi(\sigma), \quad \sigma = P^T\omega, \quad t \in I = [0,\infty), \tag{1}$$

where $x \in R^n$ is a state vector of the object, $f \in R^n$ is a vector-function, satisfying to conditions of existence of a solution $x(t) = 0$, and $B_1 \in R^{n\times r}, P \in R^{n\times r}$ are matrices, $\omega \in R^s (s \leq n)$ is a vector, $\xi \in R^r$ is a differentiable on σ vector-function, satisfying of local quadratic connection

$$\begin{gathered}\varphi(0) = 0 \wedge 0 < \sigma^T\varphi(\sigma) \leq \sigma^T K\sigma, \quad \forall \sigma \neq 0,\\ K_1 \leq \frac{\partial\varphi(\sigma)}{\partial\sigma} \leq K_2, \{K = K^T > 0\} \in R^{r\times r}, K_i = K_i^T > 0.\end{gathered} \tag{2}$$

Definition 1. The program manifold $\Omega(t)$ of automatic control system is called absolutely stable if it is globally stable on solutions of system (1) for any $\omega(t_0, x_0)$ and $\varphi(\sigma)$, satisfying conditions (2).

Differentiating ω from (2) by time t based on (1) we obtain

$$\dot{\omega} = \frac{\partial\omega}{\partial t} + Hf(t,x) - HB_1\xi, \quad \xi = \varphi(\sigma), \quad \sigma = P^T\omega, \quad t \in I = [0,\infty), \tag{16}$$

Choosing the function $F(t,x,\omega)$ in linear form with respect to ω we obtain the following system:

$$\dot{\omega}(t) = A\omega - B\xi, \quad \xi = \varphi(\sigma), \quad \sigma = P^T\omega, \quad t \in I = [0,\infty).$$

Here nonlinearity satisfies to generalized conditions (15), and $F(t,x,\omega) = -A\omega, A \in R^{s\times s}, H = \dfrac{\partial\omega}{\partial x}, B = HB_1$.

For system (3), we construct the Lyapunov function in the following form

$$V(\omega) = \omega^T L\omega, \tag{4}$$

where L is a positive-definite symmetric $s \times s$ matrix. The derivative of the function $V(\omega)$in time t by virtue of the system (17) and taking into account the condition (15) will take the form

$$-\dot{V}(\omega) = \omega^T C\omega + \sigma^T K\xi, \tag{5}$$

if the following matrix equalities are true

$$A^T l + LA = C, \tag{6}$$

$$PK = 2LB. \tag{7}$$

Theorem 1. Let a definite-positive matrix C be given, the nonlinearity $\varphi(\sigma)$ satisfies the conditions (2), there exists a matrix L. satisfying equation (6). Then for the absolute stability of the program manifold $\Omega(t)$ with respect to a given vector function ω it is enough to perform matrix equalities (6), (7).

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