PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 150, Number 11, November 2022, Pages 4575–4586
https://doi.org/10.1090/proc/15929
Article electronically published on August 18, 2022

ON INFINITE MACWILLIAMS RINGS AND MINIMAL INJECTIVITY CONDITIONS

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(Communicated by Jerzy Weyman)

Abstract. We provide a complete answer to the problem of characterizing left Artinian rings which satisfy the (left or right) MacWilliams extension theorem for linear codes, generalizing results of Iovanov [J. Pure Appl. Algebra 220 (2016), pp. 560–576] and Schneider and Zumbrägel [Proc. Amer. Math. Soc. 147 (2019), pp. 947–961] and answering the question of Schneider and Zumbragel [Proc. Amer. Math. Soc. 147 (2019), pp. 947–961]. We show that they are quasi-Frobenius rings, and are precisely the rings which are a product of a finite Frobenius ring and an infinite quasi-Frobenius ring with no non-trivial finite modules (quotients). For this, we give a more general "minimal test for injectivity" for a left Artinian ring A: we show that if every injective morphism $\Sigma_k \to A$ from the k'th socle of A extends to a morphism $A \to A$, then the ring is quasi-Frobenius. We also give a general result under which if injective maps $N \to M$ from submodules N of a module M extend to endomorphisms of M (pseudo-injectivity), then all such morphisms $N \to M$ extend (quasi-injectivity) and obtain further applications.

Introduction

Frobenius algebras have their roots in the work of Georg Frobenius, and have since surfaced in many fields of mathematics from algebra, representation theory to geometry, topology and quantum groups to name a only few. A (necessarily) finite dimensional algebra A over a field K is Frobenius if the left (equivalently, right) regular representation A is isomorphic to its K-dual A^*. The categorical (Morita invariant) version of this notion is that of quasi-Frobenius algebras, or more generally, in the absence of a ground field, quasi-Frobenius (QF) rings. These are rings A which are left (or equivalently, right) Artinian and injective as a left (equivalently right) module over themselves. This class of rings and algebras has been studied extensively in literature, by many authors (see, for example, [NY] or [K] and references therein). More recently, Frobenius rings became very important in coding theory, once the study of codes over finite rings other than $\mathbb{F}_2$ emerged. The main reason for this has to do with MacWilliams' extension theorem, proved initially by F.J. MacWilliams in [M] for linear codes over $\mathbb{F}_2$: the class of finite Frobenius rings consists of precisely the finite rings for which the MacWilliams extension theorem for linear codes holds (see below). This very interesting and important connection

Received by the editors January 17, 2019, and, in revised form, December 12, 2020.
2020 *Mathematics Subject Classification.* Primary 16D50, 16P20, 94B05, 11T71.
Key words and phrases. Linear codes, MacWilliams theorem, finite ring, quasi-injective, pseudo-injective, quasi-Frobenius, QF ring.

was proved by J.A. Wood in his fundamental work [W'99, W'08], with contributions to the theory by several other authors [DL-P1'04, DL-P2'04, GS'00, WW'96] (see also [I'16] for an account).

If A is a ring, a *linear code* over A is simply an A-submodule L of A^n. The weight $wt(x)$ of an element $x = (x_1, \ldots, x_n) \in A^n$ is defined as the number of non-zero entries $x_i \neq 0$ in x. A monomial transformation of A^n is a map $f : A^n \to A^n$ defined by $f(x_1, \ldots, x_n) = (x_{\sigma(1)}u_1, \ldots, x_{\sigma(n)}u_n)$ where σ is a permutation of $\{1, \ldots, n\}$ and $u_i \in U(A)$ are invertible. MacWilliams' original theorem for linear codes states that every injective linear map $f : L \to \mathbb{F}_2^n$ from a linear subpace L of $\mathbb{F}_2^n$ extends to a monomial transformation of $\mathbb{F}_2^n$. By results of J.A. Wood, the same holds for any linear code L and any injective morphism of A-modules $f : L \to A^n$, provided A is a finite Frobenius ring, and furthermore, as noted before, Frobenius rings are exactly the class of finite rings where this happens.

MacWilliams' extension property, as a property related to extending morphisms, is naturally closely related to various (weak) self-injectivity properties of the ring; the connection between this property and QF-rings in general and beyond the context of finite rings was recently studied in [I'16, SZ'17]. We say that a ring is *left MacWilliams* [SZ'17] if MacWilliams' extension theorem holds for left submodules L of A^n, for all n. The result of Wood was first extended to Artin algebras in [I'16], where it was proved that an Artin algebra which is a left MacWilliams ring is a product of a finite Frobenius ring and an infinite quasi-Frobenius ring which has no non-trivial finite modules; moreover, conversely, such Artin algebras are both left and right MacWilliams. The results of [I'16] also unified some results of Honold [Ho'01] and [DL-P1'04, DL-P2'04] on finite Frobenius rings and some classical characterization results for Frobenius algebras over fields of Nakayama from his foundational series [N'39, N'41, N'43, N'49]. Nevertheless, one indispensable ingredient of the treatment over Artin algebras - a common feature of finite rings and algebras over fields - was the presence of the underlying category of k-modules and the existence of k-duality over a ground commutative Artinian ring k. Remarkably, an extension of these to the case of arbitrary Artinian rings was recently also proved in [SZ'17]; it is shown there that a left Artinian ring A is left MacWilliams if it is left pseudo-injective (see below) and the finitary socle of A (=the sum of the finite simple A-submodules of A) embeds in $A/J(A)$ [SZ'17, Theorem 4.7], and one can regard this as a "Finitary Frobenius" property. The question of fully characterizing arbitrary left Artinian left MacWilliams rings, and whether they are always QF, thus naturally arises and is stated in [SZ'17].

One goal and main result of this paper is to provide a complete positive answer to this question, and settle the problem of characterizing left Artinian left MacWilliams rings. On the other hand, several related notions exist in literature and have been studied by many authors. That is, note that the MacWilliams extension condition for $n = 1$ simply asks that injective maps $f : I \to A$ from left ideals of A extend to module automorphisms of A (i.e. are given by right multiplication by a unit). More generally, a module M over a ring A is said to be *quasi-injective* if every map $f : L \to M$ from a submodule L of M extends to an endomorphism of M, and M is said to be *pseudo-injective* if every such injective map $f : L \to M$ extends to an endomorphism of M. Hence, in this language, a left MacWilliams ring is left pseudo-injective. Recent work of [ESS'13] proved a very interesting connection: a module is pseudo-injective if and only if it is *auto-invariant*, which means M is invariant under automorphisms of its injective envelope $M \subseteq E(M)$.

The question of whether pseudo-injective implies quasi-injective for modules has been considered by many authors, starting with work of [DF'69], and also recently, with a general theory and related notions developed in [AFT'15], [ESS'13], [GS'14], [GTS'15]. This question is also closely related to the problem of writing endomorphisms (or elements of rings) as sums of units [KS'07] (see also [LMPZ'15], [EI'19]). A ring is quasi-injective if and only if it is (self)injective (by Baer's criterion), but a ring may be pseudo-injective and not self-injective [GTS'15]. But, whenever pseudo-injectivity implies injectivity, it provides a smaller set of extension conditions to check. Thus, a general question of interest is the following

Question 0.1. Given a class of rings $\mathcal{C}$, find minimalistic sets of extension conditions that need to be satisfied for a ring A in $\mathcal{C}$ in order to ensure that A is self-injective; that is, determine "minimal" sets of maps $S_A \subset \{f|f : I \to A; I \text{ left ideal of } A\}$ such that if every morphism in S_A has an extension to A, then A is left (self)injective.

Besides pseudo-injectivity, such minimal or weak injectivity conditions have also been considered before, for example, in the form of *mininjective rings* [Ha'82], [Ha'83], [NY'97], or *principally injective rings* (p-injective rings; see also [NY]). A ring is left mininjective, respectively, p-injective, if every morphism $V \to A$ from a simple, respectively, principal, left ideal V of A extends to an endomorphism of A (i.e. is given by right multiplication by some $a \in A$). It is also known that even for Artinian rings, neither mininjectivity nor p-injectivity implies self-injectivity (by an example of Bjork; see Example 1.3). Our main result here is to give such a minimal injectivity test, showing that for a left Artinian ring, to obtain self-injectivity it is always enough to check the extension condition for injective morphisms from a certain finite set of ideals of A; more precisely, we prove:

Theorem 0.2. *Let A be a left Artinian ring, and $\Sigma_0 \subset \Sigma_1 \subset \dots$ be the left Loewy series of A. If every injective morphism $f : \Sigma_k \to A$ has an extension $\overline{f} : A \to A$, then A is a QF ring.*

Moreover, using Bjork's example, we show that this theorem is indeed a type of minimal test for injectivity (see Remark 1.8). Related results were known before but only for particular classes of Artinian rings; it is proved in [I'16] that for Artin algebras, mininjectivity implies QF, a fact which was previously known for finite dimensional algebras (see [NY'97, NY] and the classical work of Nakayama) and for finite rings [Ho'01] (as noted above, it is not true for Artinian rings in general).

Corollary 0.3 is a consequence of this theorem, and combined with a decomposition result of [I'16], it also answers the above mentioned question on MacWilliams rings.

Corollary 0.3. *Let A be a left Artinian ring. If A is left pseudo-injective, then A is QF. In particular, a left Artinian left MacWilliams ring is QF and hence also right Artinian (and thus right MacWilliams), and it decomposes as a product of a finite Frobenius ring and a QF ring with no non-trivial finite modules.*

We also give a second proof of the above corollary, based on general results of [GS'14, GTS'15] and [AFT'15] which extend classical work of [AF] and build up on work of [KS'07, ESS'13]; these results show that under mild conditions, pseudo-injectivity implies quasi-injectivity. This second proof has the advantage of being very short, but requires the assumption that the ring has no quotients isomorphic to $\mathbb{F}_2$; as this is based on the body of work of the aforementioned papers, we also provide a short independent proof that applies directly to our situation. For this, we show that if M is a pseudo-injective module with essential socle M_0 which has no isotypical component of cardinality 2, then M is quasi-injective. In particular, this also applies to show that a left Artinian right MacWilliams ring is also QF except for possibly few exceptions.

Corollary 0.4. *Let A be a left Artinian, right pseudo-injective ring, which does not have $\mathbb{F}_2$ as a quotient. Then A is QF. In particular, a left Artinian, right MacWilliams ring which does not have $\mathbb{F}_2$ as a quotient is QF (and also left MacWilliams) and has a decomposition as above into a product of a finite Frobenius ring and a QF-ring with no finite quotients.*

1. The result

We begin by recalling some well known standard terminology, and to fix notation. Let A be a ring. Given an A-module M, the socle of M (= sum of all simple submodules) is denoted by $\operatorname{soc}(M)$, and $J(M)$ will denote the Jacobson radical of M. If M has finite length, we denote $\operatorname{top}(M) = M/J(M)$; this is a semisimple module, which is the largest semisimple quotient of M. Given a ring A, a left A-module M, and right A-module N, we will denote ${}^*M = \operatorname{Hom}_A(M, A)$ the (left) A-dual of M (which is a right A-module) and by $N^* = \operatorname{Hom}(N, A)$ the (right) A-dual of N (which is a left A-module). This will avoid confusion when we consider A-duals of certain bimodules. We will denote by $\mathcal{S}$ a set of representatives for simple left A-modules. If S is a simple A-module, we will let $P(S)$ denote its projective cover.

It was originally proved in [NY'97] that a left Artinian ring is left mininjective precisely when the A-dual ${}^*S = \operatorname{Hom}_A(S, A)$ of any simple left module is either simple or 0 (see also [Ha'82]). This was slightly strengthened in [I'16], and Lemma 1.1 summarizes results of that paper on mininjectivity that will be needed here; part (i) below is [I'16, Proposition 3.8], part (ii) is [I'16, Theorem 3.12(a)] and part (iii) is [I'16, Proposition 3.7] and [Ha'82, Theorem 5].

Lemma 1.1. *Let A be a left Artinian. Then the following hold:*

(i) *A is a left mininjective ring if and only if for every simple left module S, the right module ${}^*S$ is a simple module.*

Furthermore, in the case when A is left Artinian left mininjective then:

(ii) *A is also right Artinian.*

(iii) *Then there exists a permutation $\tau(S)_{S\in\mathcal{S}}$ of the simple left modules such that for each $S \in \mathcal{S}$, $soc(P(S)) = \tau(S)^{n(S)}$ for some number $n(S) \geq 1$.*

We will also need the following result from [I'16], which provides a decomposition theorem for Artinian rings.

Lemma 1.2 ([I'16, Corollary 2.3])**.** *Let A be a left and right Artinian ring. Then A decomposes as a direct product of rings $A = \prod_{i=1}^{n} A_i$ where for each i, the ring A_i is either finite or it has the property that all of its simple modules have the same infinite cardinality.*

By the celebrated criterion of Baer, A is left self-injective if the extension property holds for maps $I \to A$ for all left ideals of A. In what follows, we provide a result which answers the question of finding a minimal set of extension conditions that need to be verified in order to ensure (self)injectivity of A. That is, we are looking for a set of left ideals $\mathcal{S}$ of a ring A which is as small as possible, and such that if every map $f : I \to A$ for $I \in \mathcal{S}$ extends to A, then A is left self-injective. By [I'16, Theorem 3.11], for Artin algebras (and hence, finite dimensional algebras over fields and finite rings), it is enough to check the extension property for *simple* ideals. We note though that, by Björk's example [Bj'70] below (see also [NY, Example 2.5, 38]), it is possible to have a ring which is left Artinian and left mininjective (and even p-injective, which means the extension property holds for all principal ideals), but not QF.

Example 1.3 (Björk)**.** Let $K = F(X)$ be the field of rational functions in one variable over the field F, and consider the quotient $R = K[T]/(T^2)$ as an F-vector space. Modify the ring structure of the F-vector space R as follows: write $R = K \oplus TK$ as a direct sum of right K-spaces. On the F-subspace TK, let K act as usual to the right, but modify the left action of $K = F(X)$ via the embedding $\sigma : F(X) \to F(X)$ which takes X to X^2. That is, let R by the semitrivial extension of K with the K-bimodule ${}_{\sigma(K)}K_K$, which is simply just K with the usual right K-action, and left K action given via σ. Specifically, the multiplication of two elements $a + Tb, c + Td \in R$ $(a, b, c, d \in K)$ is given by

$$(a + Tb)(c + Td) = ac + T(bc + \sigma(a)d).$$

This ring A is local and right uniserial, and $Soc({}_AA)$ has length 2. It is thus left and right Artinian, and left mininjective (and left p-injective), but not QF. We note that on the left, the left extension property $I \to A$ fails only for one ideal, namely, for $I = Soc(A)$, i.e. for maps $f : Soc(A) \to A$.

Note that if I is a left ideal of a ring A, the extension property for maps $f : I \to A$ is satisfied if and only if the sequence $0 \to {}^*(A/I) \to {}^*A \to {}^*I \to 0$ is exact, or simply the map ${}^*A \to {}^*I$ is surjective. Hence, if I is such an ideal and J is a direct summand of I, then the map ${}^*A \to {}^*J$ is also surjective as it is the composition of ${}^*A \to {}^*I \to {}^*J$, the first map being surjective and the second map being split surjective in this case.

As noted, in the case of Artin algebras, one only needs to check the extension property on simple ideals, which means that in that case one only needs to check this condition for finitely many ideals (there may be infinitely many simple ideals in A but it is enough to check the extension property for each isomorphism type of simple S and maps $S \to A$). We note that something similar holds for general Artinian rings: there is a set of finitely many ideals on which this condition can be checked to ensure injectivity of ${}_AA$. It uses ideas resembling those used by Ikeda and Dieudonné (see also [I'16, Theorem 3.12] and methods of [I'15]); it will be strengthened eventually by the main result, but it is a main step in its proof. We will denote by $\Sigma_0, \Sigma_1, \dots, \Sigma_k, \dots$ the left Loewy series of the left Artinian ring A. We recall that if M is a left A-module, its Loewy series L_α is defined for ordinals α inductively as follows: $L_0 = soc(M)$, and for arbitrary α, if $\alpha = \beta + 1$ is a successor, then L_α is such that $L_\alpha/L_\beta = soc(M/L_\beta)$, and if α is a limit ordinal, then $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$.

Lemma 1.4. *Let A be a left Artinian ring. Suppose the extension property holds for morphisms of left modules $\Sigma_k \to A$, for all k. Then A is QF.*

Proof. By the previous remark, we note that A has the extension property for maps $f : S \to A$ from simple ideals, so it is left mininjective. Thus, the dual ${}^*S$ of any simple module is simple, and A is also right Artinian, by Lemma 1.1. Using the fact that for any short exact sequence of left A-modules $0 \to X \to Y \to Z \to 0$, the dual sequence $0 \to {}^*Z \to {}^*Y \to {}^*X$ is a left exact sequence of right modules, a simple induction on length implies then that for every module Y of finite length, the dual ${}^*Y$ has finite length and $length({}^*Y) \leq length(Y)$. Consider the inclusions of left modules $\Sigma_{k-1} \hookrightarrow \Sigma_k \hookrightarrow A$; dualizing we obtain a surjective composition ${}^*A = A \to {}^*\Sigma_k \to {}^*\Sigma_{k-1}$ (since the dual ${}^*A \to {}^*\Sigma_{k-1}$ is surjective, by hypothesis). This shows that ${}^*\Sigma_k \to {}^*\Sigma_{k-1}$ is surjective, and so the sequence of right modules

$$0 \to {}^*(\Sigma_k/\Sigma_{k-1}) \to {}^*\Sigma_k \to {}^*\Sigma_{k-1} \to 0$$

is exact, and thus $length({}^*\Sigma_k) = length({}^*\Sigma_{k-1}) + length({}^*(\Sigma_k/\Sigma_{k-1}))$ (here, length is that of the right modules). Since for a semisimple left module M, we have $length(M) = length({}^*M)$ (as the dual of a left simple module is simple), another straightforward induction using this sequence shows that $length({}^*\Sigma_k) = length(\Sigma_k)$, for all k; since $\Sigma_n = A$ for some n, we get $length(A_A) = length({}^*({}_AA))$ $= length({}_AA)$ (note that $A_A = {}^*({}_AA)$).

Finally, consider now any left ideal I of A, and the exact sequence $0 \to {}^*(A/I) \to {}^*A \to {}^*I$. We have

$$\begin{aligned} length(A_A) &= length({}^*({}_AA)) \\ &\leq length({}^*(A/I)) + length({}^*I) \quad \text{(by the above)} \\ &\leq length(A/I) + length(I) \quad \text{(since } length({}^*Y) \leq length(Y)\text{)} \\ &= length({}_AA) = length(A_A). \end{aligned}$$

Thus, we must have equalities all through; in particular, $length({}^*({}_AA))$ $= length({}^*(A/I)) + length({}^*I)$ implies that the morphism ${}^*A \to {}^*I$ is surjective, which means A is left self-injective (by Baer) and so QF. □

Definition 1.5. Let A be a ring, and I a left ideal. We will say that the weak extension property holds for maps $I \to A$ (or simply for I) provided that every injective morphism $f : I \to A$ extends to a morphism $g : A \to A$ (equivalently, $f(x) = xr$ for some $r \in A$).

We also need Lemma 1.6, forms of which have been noted in literature, all with roots in the original result of Zelinsky [Z'54] and Wolfson [Wo'53] that every element in $\mathrm{End}_D(V)$ for a vector space V over a division ring D is a sum of two units except for the case when $\dim(V) = 1$ and the division ring is $\mathbb{F}_2$ (see also [LMPZ'15], [EI'19] for related results).

Lemma 1.6. *Let A be a semilocal ring which has no non-trivial finite simple modules. Then every element in A is a sum of units.*

Proof. First, write $A/J = \prod_{i=1}^n M_{n_i}(D_i)$, so each D_i is infinite, since its corresponding simple is the column space $D_i^{n_i}$. Each $x_i \in M_{n_i}(D_i)$ is then a sum of two units $x_i = u_i + w_i$ by [Z'54], so this holds for the product of matrix rings A/J. This immediately lifts mod J: if $a \in A$, write $a = u + w$ (mod J) where (the images of) $u, w \in A$ are units in A/J, and hence u, w are also units in A. Then

$a = u + w + x$, $x \in J$ and we may write $x = (x-1)+1$, and $1-x, 1 \in U(A)$, so $a = u + w + (x-1) + 1$ is a sum of units. □

The hypothesis of the statement above can be weakened to that A has no quotients isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2$ (upcoming [EI'19]), but that will not be needed here.

We will also need the following straightforward remark: if $A = R_1 \times R_2$ is a product of rings, and I is a left ideal (so $I = I_1 \times I_2$), then the (weak) extension property for I holds in A if and only if the (weak) extension property holds for I_1 and I_2 in R_1 and R_2 respectively.

Theorem 1.7 shows in particular that a left Artinian pseudo-injective ring is quasi-injective (so QF), which does not seem to have been noted before in literature.

Theorem 1.7. *Let A be a left Artinian ring. If the weak extension property holds for (injective) maps of left modules $\Sigma_k \to A$ for each $k = 0, 1, 2, \dots$, then A is a QF ring. In particular, if A is left pseudo-injective, then A is QF.*

Proof. First, note that if S is a simple left ideal, and $f : S \to A$ is a non-zero morphism, then $Im(f) \subseteq \Sigma_0$, and $S \subseteq \Sigma_0$. Clearly, since S is a direct summand of Σ_0, we may first extend f to an *injective* endomorphism (and hence, automorphism) of Σ_0 (this is just linear algebra over the division ring $\mathrm{End}(S)$). This yields a map $g : \Sigma_0 \to A$ which can be further extended to A by hypothesis (note that we cannot apply directly the remark that the extension property holds for direct summands, as we only know it holds for injective maps). This shows that the ring A is left mininjective. By Lemma 1.2, $A = A_F \times A_I$, where A_I has only infinite simple modules, and A_F is a finite ring.

Now, by the remark before, and the fact that the Loewy series is categorical, we get that A_F and A_I satisfy the same property as A; in particular, A_F is mininjective, and thus it is QF (as noted above, by [I'16, Theorem 3.12], since A_F can be regarded as an Artin algebra; this is known also from [Ho'01]). It remains only to prove that A_I is QF, and to simplify notation, we will assume in what follows that $A = A_I$ so A has no non-trivial finite modules.

We first re-interpret the weak extension property for Σ_k. Any injective map $f : \Sigma_k \to A$ must have its image contained in Σ_k (Σ_k are fully invariant), and since Σ_k has finite length, f has to be an automorphism. Also, again since Σ_k are invariant under endomorphisms of A, the restriction from A to Σ_k provides a map

$$Res_k : A^{\mathrm{op}} = \mathrm{End}_A(A) \to \mathrm{End}_A(\Sigma_k)$$

which is a morphism of rings. The hypothesis says simply that every invertible element of $\mathrm{End}_A(\Sigma_k)$ is in the image of Res (and is equivalent to this). Now, since Σ_k has finite length, $\mathrm{End}_A(\Sigma_k)$ is a semiprimary ring, and in particular it is semilocal. Also, $\mathbb{F}_2$ cannot be a quotient of $\mathrm{End}_A(\Sigma_k)$: if there exists $g : \mathrm{End}_A(\Sigma_k) \to \mathbb{F}_2$ surjective, then $f = g \circ Res_k \neq 0$ since $f(1) = g(\mathrm{Id}_{\Sigma_k}) = 1 \in \mathbb{F}_2$, and then f is automatically surjective, which contradicts the running assumption that A has no non-trivial finite quotients. Therefore, since every element of $\mathrm{End}_A(\Sigma_k)$ is a sum of units by the previous Lemma, and $U(\mathrm{End}_A(\Sigma_k)) \subseteq \mathrm{Im}(Res_k)$, it follows that Res is surjective. Translated, this simply means that the extension property is satisfied for the left ideals Σ_k. By Lemma 1.4, this implies that R is QF. □

Remark 1.8. We note that again by Bjork's example, the above Theorem is indeed a minimal test for injectivity: the ring A in that example has $\Sigma_1 = A$, and extension of maps fails only for some injective maps $\Sigma_0 \to A$.

Although included in the previous theorem, it seems worthwhile to also record Corollary 1.9 separately as another "minimal test for injectivity". It generalizes [Ha'82, Theorem 13], and results of [Ha'83] or [N'39, N'41] on finite dimensional algebras (see also [NY'97]), and of [I'16] on artin algebras.

Corollary 1.9. *A left Artinian ring is QF if and only if it is left pseudo-injective.*

2. Infinite MacWilliams rings

We can now use this to obtain the complete equivalent characterization of MacWilliams rings. Recall that a ring A is left MacWilliams if every weight preserving morphism $f : N \to A^n$ from a left submodule N of A^n extends to an automorphism of A^n. Theorem 2.1 generalizes the results of [Ho'01] for finite rings (see also [DL-P1'04, DL-P2'04]), the results of [I'16] for artin algebras, and the partial results of [SZ'17], at the same time answering the question of Schneider and Zumbrägel [SZ'17, Section 4].

Theorem 2.1. *Let A be a left Artinian ring. Then A is a left MacWilliams ring if and only if $A = A_F \times A_I$ is a product of a finite Frobenius ring and a quasi-Frobenius ring A_I which has no non-trivial finite modules. Moreover, in this case, A is also right MacWilliams.*

Proof. If the ring A has this form $A = A_F \times A_I$, then it is left and right MacWilliams for example by [I'16, Theorem 4.1] (see also [SZ'17]). Conversely, note that the extension property for linear codes of length 1, i.e. for submodules of A, implies that A is left pseudo-injective which together with the hypothesis of A left Artinian implies that A is QF by Theorem 1.7. Now, using Lemma 1.2 (or [I'16, Corollary 2.3]), we have that $A = A_F \times A_I$ where A_F is finite and A_I has no non-trivial finite modules, and all remain MacWilliams rings (in fact, one can split $A_I = \prod_i A_i$ according to cardinalities of simples). Hence, both A_F, A_I are QF, but A_F is Frobenius by Wood's results [W'99, W'08]. □

We end by noting some questions that appear here. While the natural extension of the study of finite MacWilliams rings seems to be the Artinian realm, one may ask what does this property mean for other situations, a question which seems to implicitly appear also in [SZ'17]. Hence, we state:

Question 2.2. What can be said about left (or left-right) MacWilliams rings in general? Is a Noetherian/Noetherian commutative left MacWilliams ring necessarily Artinian (and hence, QF)?

The above results, as noted, are a type of minimal test condition for injectivity. One can ask of such conditions in general for modules. Of course, Baer's criterion is very useful in checking injectivity as it reduces the problem to ideals of the ring; one may ask, nevertheless, if this can be further reduced to a smaller class of ideals, or of maps, such as the injective maps. The same question can be asked for quasi-injective (i.e. self-injective) modules. Question 2.3 is a version of a question the author learned from M. Yousif [Y'18], and stems also from work of A. Fachinni (see [AFT'15]).

Question 2.3. Let M be an Artinian module. If M is pseudo-injective, is then M necessarily quasi-injective?

A positive answer would be a generalization of the main result of this paper, which shows that this is true when the module M is the ring itself. Many sufficient conditions under which M pseudo-injective implies that M is quasi-injective have already been found by a number of authors, starting with the initial work of [DF'69], but also [GS'14, ESS'13] and references therein. It is true, for example, if the endomorphism ring of M does not have any quotients isomorphic to $\mathbb{F}_2$ [GS'14, Theorem 3]. Often, the proofs of such an implication are based on versions and generalizations of Zelinsky's result allowing one to write endomorphisms of modules as sums of automorphisms, and for that reason, one often needs to exclude the pathological case of $\mathbb{F}_2$ in some way (as above; see also [LMPZ'15] for similar situations preventing the expression of every element in an algebra as sums of units). The aforementioned result of [GS'14] is based on previous work in [KS'07] and also on characterizations of automorphism-invariant modules [ESS'13, AFT'15]. On the other hand, if such a quotient is present, examples are known where pseudo-injectivity does not imply quasi-injectivity [GTS'15, Example 3.4]. For this reason, we preferred the approach above, which also yields some more general results on Artinian rings potentially of independent interest, such as Theorem 1.7 and its immediate corollary.

We do note, however, an alternative approach of the main results, which (only) works under mild conditions which exclude the presence of $\mathbb{F}_2$ (as a quotient), and which also gives an answer to the question above in another class of examples. It will also provide additional symmetry results for MacWilliams rings. While this approach can be implemented immediately using the above mentioned [GS'14, Theorem 3], since that result is based also on some non-trivial body of previous work, we also provide a self-contained short proof based on the following.

Proposition 2.4. *Let M be a module over a ring A and suppose that $M_0 = soc(M)$, the socle of M, is an essential submodule (this is true, for example, when M is semiartinian). Suppose, in addition, that*

$(*)$ *every isotypical component of M_0 has at least 3 elements.*

If M is pseudo-injective, then it is also quasi-injective. Moreover condition $(*)$ is verified if $\operatorname{End}(M_0)$ has no quotient isomorphic to $\mathbb{F}_2$, or when $\operatorname{End}(M)$ has no such quotient.

Proof. We observe first that $(*)$ is verified as noted in the last statement. We have $M_0 = \bigoplus_S M_S$, where the sum is over isomorphism types of simple modules, and $M_S = \sum_{T\subset M;\, T\cong S} T$ is the S-isotypical component of M_0. Condition $(*)$ means that $|M_S| > 2$. Then $\operatorname{End}_A(M_0) = \prod_S \operatorname{End}_A(M_S)$, and $(*)$ is also equivalent to asking that none of the blocks $\operatorname{End}_A(M_S)$ is isomorphic to $\mathbb{F}_2$. This is verified if $\operatorname{End}_A(M_0)$ has no quotients isomorphic to $\mathbb{F}_2$. Also, since M_0 is fully invariant, there is a restriction morphism $Res : \operatorname{End}(M) \to \operatorname{End}(M_0)$, and if $\operatorname{End}(M)$ has no quotients isomorphic to $\mathbb{F}_2$, then $\operatorname{End}(M_0)$ doesn't either, since such a quotient $\operatorname{End}(M_0) \to \mathbb{F}_2$ produces as before a map $\operatorname{End}_A(M) \stackrel{Res}{\to} End_A(M_0) \to \mathbb{F}_2$ which is nonzero (takes 1 to 1) and hence surjective in this case.

Let $N \subset M$ be a submodule, and $f : N \to M$. Then $N_0 = N \cap M_0$ is also essential in N. Also, N_0 is a direct summand in M_0 and $f(N_0) \subset M_0$; let $f_0 \in \operatorname{End}_A(M_0)$ be an extension of $f|_{N_0} : N_0 \to M_0$ (which exists by semisimplicity). Each block $\operatorname{End}_A(M_S)$ is the endomorphism ring of a vector space over a division ring, and is not isomorphic to $\mathbb{F}_2$, and by results of [Wo'53, Z'54], every element is a sum of two units, which implies the same for the product. Write $f_0 = \alpha_0 + \beta_0 \in \mathrm{End}_A(M_0)$ with $\alpha_0, \beta_0 \in \mathrm{Aut}(M_0)$, and let $\alpha, \beta \in \mathrm{End}(M)$ be extensions of α_0, β_0 when considered as injective morphisms $M_0 \to M$. Then $f - \alpha|_N$ is injective. Indeed, otherwise $N_0 \cap \ker(f - \alpha|_N) \neq 0$ since N_0 is essential in N; but for $x \in N_0 \cap \ker(f - \alpha|_N)$, $f(x) = f_0(x) = \alpha(x) = \alpha_0(x)$, which implies $\beta_0(x) = f(x) - \alpha_0(x) = 0$, which implies $x = 0$. Hence, $f - \alpha|_N$ extends to $g \in \mathrm{End}_A(M)$, i.e. $f - \alpha|_N = g|_N$ so $f = (\alpha + g)|_N$. □

The result above in the case when $\mathrm{End}_A(M)$ has no quotients isomorphic to $\mathbb{F}_2$ follows directly from the results of [GS'14] and [ESS'13] (which shows pseudo-injective modules are the same as auto-invariant modules); we note that, however, hypothesis $(*)$ does not seem to immediately imply that $\mathrm{End}_A(M)$ has no such quotients. Note that even in the case of a ring $R = \prod_{i \in I} \mathrm{End}_{D_i}(V_i)$ which is product of blocks $\mathrm{End}_{D_i}(V_i)$ for vector spaces V_i over division rings D_i, R may potentially have quotients isomorphic to $\mathbb{F}_2$ without any of these blocks being equal to $\mathbb{F}_2$, as the maximal spectrum of R has a complicated structure depending on ultrafilters on I and type of blocks involved. We note that the above proposition also gives an answer to Question 2.3 in a fairly general case, but it would be interesting to know whether the requirement of condition (*), excluding a certain presence of $\mathbb{F}_2$, could be eliminated in the Artinian case.

Corollary 2.5. *Let A be a left Artinian ring which is right pseudo-injective. If A does not have any quotients isomorphic to $\mathbb{F}_2$, then A is QF.*

Proof. As noted before, one can simply apply [GS'14, Theorem 3] and note that $A = \mathrm{End}_A(A_A)$ has no quotients isomorphic to $\mathbb{F}_2$ so A_A is quasi-injective, i.e. A is right self-injective, and left Artinian, so it is QF.

Alternatively, note that $A^{op} = \mathrm{End}({}_AA)$ is a semiprimary ring (the endomorphism ring of an Artinian module), and so A is a semiprimary ring. It follows that A_A is semiartinian, and since A has no quotients isomorphic to $\mathbb{F}_2$, any simple module has at least three elements; therefore, the hypothesis of the previous proposition is satisfied and thus A_A is quasi-injective, and the proof is again finished. □

Although perhaps removing the hypotheses on $\mathbb{F}_2$ from the statement does not seem like a large improvement, it is still tempting to ask if the above Corollary can be stated in the most general form:

Question 2.6. If A is a left Artinian, right pseudo-injective ring, does it follow that A is QF?

Note that, as announced, Proposition 2.4 (or alternatively, [GS'14, Theorem 3]) also provides a direct proof for Theorem 1.7, but only for the case when $\mathbb{F}_2$ does not appear as quotient. In particular, we have the following application to MacWilliams rings.

Corollary 2.7. *If A is a left Artinian, right MacWilliams ring which does not have $\mathbb{F}_2$ as a quotient, then A is QF.*

Acknowledgments

The author would like to thank Mohamed Yousif for interesting conversations on the theme and main results of this paper, as well as bringing a few references and questions to our attention. He would also like to thank Vic Camillo for stimulating feedback on this and reference [I'16]. He is also thankful to the referee for his/her thorough reading of the paper and many detailed suggestions.

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