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Proof of a conjecture of Zahariuta concerning a problem of Kolmogorov on the ϵ-entropy

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Abstract. We prove a conjecture of Zahariuta which itself solves a problem of Kolmogorov on the ϵ-entropy of some classes of analytic functions. For a given holomorphically convex compact subset K in a pseudoconvex domain D in $\mathbf{C}^n$, Zahariuta's conjecture consists in approximating the relative extremal function $u^*_{K,D}$, uniformly on any compact subset of $D \setminus K$, by pluricomplex Green functions on D with logarithmic poles in the compact subset K.

Contents

1. Introduction and statement of results

1.1. Zahariuta's conjecture

In one complex variable, potentials play at least two roles. On the one hand they provide an important source of examples of subharmonic functions. On the other hand, despite their apparently rather special nature we know that potentials turn out to be almost as general as arbitrary subharmonic functions. Indeed, we have the Poisson-Jensen formula: if D is a bounded domain in $\mathbf{C}$ where we can solve the Dirichlet problem, and if u is a subharmonic function on a neighbourhood of $\bar{D}$, then

$$u(z) = \int_{\partial D} u(w) d\omega_D(z, w) - \frac{1}{2\pi} \int_D \tilde{g}_D(z, w) \Delta u(w), \ \forall z \in D.$$

Here ω_D denotes the harmonic measure for D and $\tilde{g}_D(z, .)$ denotes the Green function for D with pole at z [Ran95]. As a consequence, any subharmonic function u on D which tends to 0 on the boundary, can be approximated by subharmonic functions on D of the form $\sum_{j=1}^N c_j \tilde{g}_D(z_j, .)$, where $c_j < 0$ and $z_j \in D$ (to prove this result we also use the symmetry property of the Green function with respect to the variable and the pole). Let us illustrate this with an example: let D be a bounded domain in $\mathbf{C}$ containing a compact subset K such that we can solve the Dirichlet problem on $D \setminus K$. Denote by $u_{K,D}$ the solution of the Dirichlet problem with $\phi = 0$ on ∂D and $\phi = -1$ on ∂K. If we set $u_{K,D} = -1$ on K, $u_{K,D}$ is subharmonic on D, continuous on $\bar{D}$ and harmonic on $D \setminus K$. Then $u_{K,D}$ can be uniformly approximated on any compact subset of $\bar{D} \setminus K$ by subharmonic functions on D of the type $\sum_{j=1}^N c_j \tilde{g}_D(z_j, .)$, where $c_j < 0$ and $z_j \in K$. Moreover, Δu, which is a positive measure supported on K, is approximated by a finite sum, $-2\pi \sum_{j=1}^N c_j \delta_{z_j}$, of Dirac measures. A precise version of this result was proved by Skiba and Zahariuta[1] in [SZ76].

This function $\sum_{j=1}^N c_j \tilde{g}_D(z_j, .)$ is in fact the unique solution of the following Dirichlet problem:

$$\begin{cases} u \text{ subharmonic and negative on } D, \text{ continuous on } \bar{D}, \\ \Delta u = 0 \text{ on } D \setminus \{z_1, \ldots, z_N\}, \\ u(z) = -c_j \log |z - z_j| + \text{Ø}(1) \text{ as } z \to z_j, \ \forall j = 1, \ldots, N, \\ u(z) \to 0 \text{ as } z \to \partial D. \end{cases}$$

In this case, u verifies $\Delta u = -2\pi \sum_{j=1}^N c_j \delta_{z_j}$ on D. If we denote by P the finite set $\{(z_j, -c_j), 1 \leq j \leq N\}$ where z_j are distinct points in D and $-c_j$ are positive weights, then we can set $g_D(P, .) = \sum_{j=1}^N c_j \tilde{g}_D(z_j, .)$. We call this the (negative) Green function on D with poles in P.

It is in this context that Zahariuta formulated in the 80's what is usually called Zahariuta's conjecture:

Zahariuta's conjecture. *For a given holomorphically convex compact subset K in a pseudoconvex domain D in $\mathbf{C}^n$, the relative extremal function $u_{K,D}^*$ can be approximated, uniformly on any compact subset of $D \setminus K$, by pluricomplex Green functions on D with logarithmic poles in the compact subset K.*

[1] V.P. Zahariuta asked the author to write his name in this way.

This was proved by Skiba and Zahariuta in the one dimensional case [SZ76], but up to now, this conjecture was open in the multidimensional case. The aim of this paper is to prove it in any dimension and in the most general context possible.

If D is an open set in $\mathbf{C}^n$ and E is a subset of D, the *relative extremal function* for E in D (see [Zah77], [Bed80a], [Kli81], [Kli82], [Sic81], [Sad81], [BT82]) is defined as

$$u_{E,D}(z) = \sup\{v(z) : v \text{ is psh on } D,\ v\,|_E \leq -1,\ v \leq 0\},\ z \in D. \quad (1.1)$$

We write psh for plurisubharmonic. The upper semicontinuous regularization $u^*_{E,D}$ is plurisubharmonic on D. In one variable, $u^*_{E,D}$ is closely related to the notion of harmonic measure. In several variables, a natural context for the study of this function is the class of hyperconvex domains [Ste74].

A domain D in $\mathbf{C}^n$ is *hyperconvex* if there exists a continuous plurisubharmonic exhaustion function $\varrho : D \to]-\infty, 0[$. If D is an open set and E is a non pluripolar relatively compact subset of D, then D is hyperconvex if and only if for any point $w \in \partial D$, $\lim\limits_{z\to w} u_{E,D}(z) = 0$.

If D is a bounded hyperconvex open set and $K \subset D$ is a compact set, then we say that K is *regular* if $u^*_{K,D}$ is a continuous function.

In one complex variable, $u_{K,D}$ is harmonic on $D \setminus K$ and $\Delta u_{K,D}$ is a positive measure supported on K. In several variables, if D is a hyperconvex domain in $\mathbf{C}^n$ containing a compact subset K, then $u^*_{K,D}$ is maximal on $D \setminus K$.

The notion of maximality in the realm of several complex variables bears the same relation to the Monge-Ampère operator as the Laplacian does in one variable. Following Sadullaev [Sad81], a plurisubharmonic function u on D is *maximal* if for every relatively compact open subset ω of D and for each upper semicontinuous function v on $\bar{\omega}$ such that v is plurisubharmonic on ω and $v \leq u$ on $\partial\omega$, we have $v \leq u$ on ω.

In one dimensional potential theory harmonic functions are smooth and are characterized in terms of the Laplace operator. As there exist discontinuous psh functions which are maximal in open sets in $\mathbf{C}^n$ with $n > 1$, the situation is quite different; if a differentiable operator is used to characterize maximal psh functions, it must be understood in some generalized (e.g. distributional) sense. The *complex Monge-Ampère operator* is a good candidate. It is defined as the nth exterior power of $dd^c = 2i\partial\bar{\partial}$, i.e. $(dd^c)^n = dd^c \wedge \ldots \wedge dd^c$ (n times). $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$. If $u \in \mathcal{C}^2(D)$, then $(dd^c u)^n = 4^n n! \det\left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right] dV$, where $dV = (\frac{i}{2})^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n$ is the usual volume form in $\mathbf{C}^n$.

Bedford and Taylor have proved ([BT76], [BT82]) that this wedge product $(dd^c u)^n$ can be defined if $u \in L^\infty_{loc}(D) \cap PSH(D)$ and u is maximal on D if and only if $(dd^c u)^n = 0$ on D.

Consequently $(dd^c u^*_{K,D})^n = 0$ on $D \setminus K$ and $(dd^c u^*_{K,D})^n$ is a positive measure supported on K.

The *pluricomplex Green functions with logarithmic poles* (see [Lem81] for strictly convex domains, [Kli85], [Dem85] and [Dem87] for hyperconvex domains, [Lel87] and [Lel89] for Banach spaces) generalize the one-variable Green functions with logarithmic poles. If D is a domain in $\mathbf{C}^n$ and P is a finite set $\{(p_j, c_j),\ p_j \in D, c_j > 0, 1 \leq j \leq N\}$, where p_j are distinct points in D and c_j are positive weights, the pluricomplex Green function on D with poles in P is defined by

$$\begin{aligned} g_D(P, z) = \sup\{v(z) : v \text{ psh on } D,\ v \leq 0, \forall j = 1, \dots, N \\ v(z) \leq c_j \log \| z - p_j \| + \emptyset(1)\}. \end{aligned} \tag{1.2}$$

If D is bounded $g_D(P, .)$ is a plurisubharmonic function on D with logarithmic poles at p_j of weight c_j, for $j = 1, \dots, N$ and $\sum_{j=1}^{N} c_j g_D(p_j, z) \leq g_D(P, z) \leq \min_j c_j g_D(p_j, z)$. Equality never holds everywhere in D, for $n \geq 2$ and $N \geq 2$, because the Monge-Ampère operator is not linear. If D is a bounded hyperconvex domain in $\mathbf{C}^n$, we have an alternative description of the pluricomplex Green functions in terms of the complex Monge-Ampère operator, namely $g_D(P, z)$ is the unique solution to the following Dirichlet problem:

$$\begin{cases} u \text{ plurisubharmonic and negative on } D, \text{ continuous on } \bar{D}, \\ (dd^c u)^n = 0 \text{ on } D \setminus \{p_1, \dots, p_N\}, \\ u(z) = c_j \log \|z - p_j\| + \emptyset(1) \text{ as } z \to p_j,\ \forall j = 1, \dots, N, \\ u(z) \to 0 \text{ as } z \to \partial D. \end{cases}$$

In this case, $(dd^c u)^n = (2\pi)^n \sum_{j=1}^{N} c_j^n \delta_{p_j}$ in D (δ_{p_j} denotes the Dirac measure at p_j).

Now we understand that a natural framework for Zahariuta's conjecture for $n > 1$ is the case where D is bounded, hyperconvex and K is regular. Indeed in this context, the functions $u_{K,D}$ and $g_D(P, .)$ are continuous on $\bar{D}$, equal to 0 on ∂D, and we can expect to obtain uniform approximations on any compact subsets of $D \setminus K$.

To formulate the main results of this paper, we need more notation. Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ containing a regular compact subset K. For any real number $c \leq 0$, $D(c)$ is the hyperconvex open subset of D defined by

$$D(c) = \{z \in D : u_{K,D}(z) < c\}. \tag{1.3}$$

For $\delta > 0$ sufficiently small, $\overline{D(-1+\delta)}$ is a holomorphically convex regular compact subset of D. We remark that $D(-1+\delta)$ is an external exhaustion of $\hat{K}_D$, the holomorphically convex hull of K in D, and that $D(-\delta)$ is an internal exhaustion of D.

We will say that a bounded domain D in $\mathbf{C}^n$ is *strictly hyperconvex* if there exist a bounded domain Ω and an exhaustion function $\varrho \in \mathcal{C}(\Omega,]-\infty, 1[) \cap PSH(\Omega)$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$ and for all real numbers $c \in [0, 1]$, the open set $\{z \in \Omega : \varrho(z) < c\}$ is connected. We will say also that a regular compact set K in a domain D in $\mathbf{C}^n$ is *strictly regular* if K is the closure of a relatively compact open subset ω in D such that $u_{K,D} \equiv u^*_{\omega,D}$.

Theorem A. *Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ containing a regular compact set K in D. Then for any $\epsilon > 0$ and $\delta > 0$ sufficiently small, there exists a pluricomplex Green function g on D with a finite number of logarithmic poles such that*

(i) the poles of g lie in the open neighborhood $D(-1+\delta)$ of $\hat{K}_D$,
(ii) g satisfies the following uniform estimates on $\overline{D} \setminus D(-1+\delta)$

$$(1+\epsilon)g(z) \leq u_{K,D}(z) \leq (1-\epsilon)g(z).$$

Theorem B. *Let D be a strictly hyperconvex domain in $\mathbf{C}^n$ containing a strictly regular compact set K in D. Then for any $\epsilon > 0$ sufficiently small, there exists a pluricomplex Green function g on D with a finite number of logarithmic poles such that*

(i) the poles of g lie in $(\hat{K}_D)^\circ$, the interior of the compact subset $\hat{K}_D$ in D,
(ii) g satisfies the following uniform estimates on $\overline{D} \setminus (\hat{K}_D)^\circ$

$$(1+\epsilon)g(z) \leq u_{K,D}(z) \leq (1-\epsilon)g(z).$$

The method used in [SZ76] to prove Zahariuta's conjecture in the one dimensional case cannot be generalized to the multidimensional case. In particular, the authors use the linearity of the Laplace operator, the Green formula in terms of the classical Green function, and the symmetry property of the Green function with respect to the variable and the pole. The problem in several variables is in particular that the complex Monge-Ampère operator is no longer linear and is not continuous with respect to the weak convergence of plurisubharmonic functions [Ceg83], the Lelong-Jensen formula contains two terms which are difficult to control, and the pluricomplex Green function with one logarithmic pole is no longer, in general, symmetric with respect to the variable and the pole.

1.2. Kolmogorov's problem

In approximation theory, instead of considering methods of approximation to specific functions by polynomials or rational functions, Kolmogorov has considered the best methods of approximation to classes of functions. That is, the classical methods (algebraic polynomials, rational functions, trigonometric polynomials) are compared with any other means of approximation from a given family of approximating sets described by the same number of parameters. This led Kolmogorov to introduce new concepts: *diameter and entropy*.

In 1936, Kolmogorov introduced the quantity called the Kolmogorov n-diameter. Let X be a normed linear space and let $C \subset X$ be a compact subset; the *Kolmogorov n-diameter* of C in X is the quantity

$$d_n(C, X) = \inf \sup_{x \in C} \inf_{y \in L_n} \| x - y \|,$$

where the lower bound is taken over all n-dimensional subspaces L_n of X.

In the 1950's, new reasons arose for Kolmogorov to return to approximation theory again and to introduce a new concept of ϵ-entropy.

On the one hand there is his study of Vitushkin's work on Hilbert's 13th problem about the complexity of function spaces. In 1955, Vitushkin proved that there are "more" functions of n variables with smoothness r than there are functions of m variables with smoothness l if $n/r > m/l$. Revealing the "entropy" meaning of this result, Kolmogorov concentrated on the Hilbert problem, and his efforts, complemented by Arnol'd's, eventually led to a refutation of Hilbert's conjecture: continuous functions of three variables did not turn out to be structured in a more complex way than functions of two variables. However, in 1958 Kolmogorov proved that the space of analytic functions of n variables is "larger" than the space of analytic functions of m variables when $n > m$.

On the other hand, Kolmogorov was enthusiastic about Shannon's information theory (1948). To distinguish a definite element in a finite set C of $N(C)$ elements, it suffices to specify $[\log_2 N(C)] + 1$ "binary digits." In the case of infinite sets, if (X, d) is a metric space and $C \subset X$ is a compact subset, Kolmogorov introduced the concept of approximate (to within ϵ) specification of an element $x \in C$ by saying that x belongs to a definite set C_i in some covering $C = \cup_i C_i$ by sets of diameter not greater than 2ϵ. The smallest cardinality of such a covering is denoted by $N_\epsilon(C, X)$. The quantity $\log_2 N_\epsilon(C, X)$ is called the *ϵ-entropy* of the set C and it is denoted by

$$H_\epsilon(C, X) = \log_2 N_\epsilon(C, X).$$

For example, in any space X of finite dimension m, the ϵ-entropy of any compact subset C of X verifies: $H_\epsilon(C, X) \log_2^{-1}(1/\epsilon)$ tends to m when ϵ tends to 0.

It is easy to see the analogy between ϵ-entropy and n-diameter. The function $N \to \epsilon_N(C, X)$, inverse to $\epsilon \to N_\epsilon(C, X)$, can be regarded as the accuracy in reconstructing an element of C when the coding is by a set consisting of N elements. The n-diameters d_n are related to the study of the approximation properties of n-dimensional subspaces, while the diameters ϵ_n are concerned with the approximation properties of sets of n points.

The determination of the entropy and diameters of function classes has several goals. Firstly, it leads to invariants, enabling us to distinguish and classify function sets in infinite dimensional spaces. Secondly, computations of diameters and entropy make it possible to find new methods of approximation. Thirdly, this is of interest for computational mathematics by giving directions for the creation of the most expedient algorithms for solving practical problems.

Precise references and details about this subject can be found in [Tik60], [KT61], [Mit61], [Tik63], [Tik83], [Kol85], [Tik89], [Tik90].

In 1956, Kolmogorov [Kol56] found the order of the ϵ-entropy of functions of n variables defined on a bounded domain in IR^n which extend analytically to some domain in $\mathbf{C}^n$: $(\log(1/\epsilon))^{n+1}$. The determination of the precise asymptotic behaviour of this ϵ-entropy comes down to the following problem: to prove the existence and to calculate explicitly the limit

$$\lim_{\epsilon\to 0} H_\epsilon\left(\mathcal{A}_K^D\right)/\log_2^{n+1}(1/\epsilon),$$

where D is a domain in $\mathbf{C}^n$ containing a compact set K, and $\mathcal{A}_K^D$ is the set of functions that are analytic in D and satisfy the inequality $||f||_D \leq 1$, endowed with the norm $||f||_K = \sup_{z\in K} \mid f(z) \mid$.

In the one dimensional case, this problem is solved (see [Vit61], [Bab58], [Ero58], [LT68], [Wid72] and see also [Far84], [Zah67], [Ngu72], [SZ76], [Ski79]): Let D be a domain in $\mathbf{C}$ and K be a compact subset in D such that we can solve the Dirichlet problem on $D \setminus K$. Let $u_{K,D}$ be the relative extremal function for K in D. Let Γ be a system of smooth contours separating K from ∂D, and let n be the normal to Γ directed from K to ∂D. The capacity associated to the compact K relative to the domain D is defined by $C(K, D) = \int_\Gamma \partial_n u(z) \mid dz \mid$. If ∂D have positive logarithmic capacity, and $\mathbf{C} \setminus D$ have a countable set of connected components, then

$$\lim_{\epsilon\to 0} H_\epsilon\left(\mathcal{A}_K^D\right)/\log_2^2(1/\epsilon) = C(K, D)/(2\pi).$$

Up to now, in the multidimensional case, this problem proposed by Kolmogorov, was open. There are some interesting publications of Zahariuta ([Zah85] and [Zah94]) on this topic. Today, this problem, which we call Kolmogorov's problem, can be precisely written as follows.

Kolmogorov's problem. $$\lim_{\epsilon\to 0} \frac{H_\epsilon\left(\mathcal{A}_K^D\right)}{\log_2^{n+1}(1/\epsilon)} = C(K, D)/(2\pi)^n \ ?$$

The *(relative) capacity of K (in D)* ([Bed80b], [BT82]) is defined by $C(K, D) = \sup\{\int_K (dd^c u)^n : u \in PSH(D, (0, 1))\}$. Note that, according to the Chern-Levine-Nirenberg estimate, $C(K, D) < \infty$. When D is a hyperconvex domain in $\mathbf{C}^n$ and K is a compact subset of D

$$C(K, D) = \int_D \left(dd^c u^*_{K,D}\right)^n = \int_K \left(dd^c u^*_{K,D}\right)^n,$$

where $u^*_{K,D}$ is the upper semicontinuous regularization of $u_{K,D}$, the relative extremal function defined in (1.1). We remark that for $n = 1$, $dd^c = \Delta dx \wedge dy$ in $\mathbf{R}^2$, and according to Green's formula $C(K, D) = \int_\Gamma \partial_n u(z) dz = \int_K \Delta u dx dy$, as it has been defined above.

Zahariuta's conjecture is in direct connection with Kolmogorov's problem. Skiba and Zahariuta have proved for $n = 1$ in [SZ76] and Zahariuta has proved for $n \geq 2$ in [Zah85] that to solve Kolmogorov's problem, it is sufficient to prove that Zahariuta's conjecture is true.

Zahariuta used methods of the theory of Hilbert spaces, in particular Hilbert scales, method of extendable bases and properties of extremal psh functions with isolated singularities, to reduce Kolmogorov's problem to his conjecture.

1.3. Statement of results

The goal of this article is to prove Zahariuta's conjecture in the multidimensional case, in the most general context possible. This will imply that Kolmogorov's problem is also solved in this context.

Suppose for the moment that D is a strictly hyperconvex domain in $\mathbf{C}^n$ (i.e. there exists a bounded domain Ω and an exhaustion function $\varrho \in \mathcal{C}(\Omega,]-\infty, 1[) \cap PSH(\Omega)$ such that $D = \{z \in \Omega : \varrho(z) < 0\}$ and for all real numbers $c \in [0, 1]$, the open set $\{z \in \Omega : \varrho(z) < c\}$ is connected) containing a regular compact set K. The relative extremal function $u_{K,D}$ is continuous on $\overline{D}$. Denote by D_j the bounded hyperconvex domain defined by

$$D_j = \{z \in \Omega : \varrho(z) < 1/j\}, \tag{1.4}$$

containing D for any integer $j \geq 1$. A proof of Zahariuta's conjecture in this case is given in the following four steps. In the fifth and last step, we will generalize Zahariuta's conjecture to the case where D is only hyperconvex.

First step. Precise version of Lelong and Bremermann's Theorem for the relative extremal function $u_{K,D}$ (part 2). If D is a domain in $\mathbf{C}^n$ containing a compact set K and p is a positive integer, we can define a compact set $\mathcal{E}_p$ of $\mathcal{O}(D)$ (the Fréchet space of holomorphic functions in D, with the topology of uniform convergence on every compact set of D) by

$$\mathcal{E}_p = \left\{ f \in \mathcal{O}(D) : \sup_{z\in D} \mid f(z) \mid \leq 1, \ \sup_{z\in K} \mid f(z) \mid \leq e^{-p} \right\} \tag{1.5}$$

and a continuous and plurisubharmonic function h_p on D by

$$h_p(z) = \sup_{f\in\mathcal{E}_p} \frac{1}{p} \log \mid f(z) \mid, z \in D. \tag{1.6}$$

Theorem 1. *If D is a strictly hyperconvex domain in $\mathbf{C}^n$ containing a regular compact set K, then*

$$\lim_{p\to\infty} h_p(z) = \sup_{p\geq 1} h_p(z) = u_{K,D}(z), \forall z \in D.$$

Second step. Approach externally $\hat{K}_D$ and internally D by special holomorphic polyhedra defined by the same n holomorphic functions (part 3). As a consequence of Theorem 1, we obtain:

Corollary 2. *For any $\epsilon > 0$, there exist three integers $j \geq 2$, $p \geq 2$ and $N = N(\epsilon) \geq 1$ and there exist N holomorphic functions $f_1, \ldots, f_N \in \mathcal{E}_{p,j} = \{f \in \mathcal{O}(D_j) :\| f \|_{D_j} \leq 1, \| f \|_K \leq e^{-p}\}$ such that $u_{K,D}(z) - \epsilon/3 \leq u_{K,D_j}(z) \leq u_{K,D}(z)$ on $\overline{D}$, and*

$$u_{K,D_j}(z) - 2\epsilon/3 \leq \sup_{1\leq l\leq N} \frac{1}{p} \log \mid f_l(z) \mid \leq u_{K,D_j}(z) \text{ on } \overline{D_{2j}}.$$

Then we decide to abandon the uniform approximation of $u_{K,D}$ by a sup of a finite (possibly large) number of plurisubharmonic functions of the type $\frac{1}{p} \log \mid f \mid$ (where f is holomorphic in a neighborhood of $\overline{D}$), in order to obtain a "good" approximation of the sets $\hat{K}_D$ and D: we approximate $\hat{K}_D$ externally and D internally by two special holomorphic polyhedra defined by the same n holomorphic functions in a neighborhood of $\overline{D}$. To do so we use an idea of Bishop [Bis61].

If $N(\epsilon^2/2) > n$ in Corollary 2, we modify the mapping $f = (f_1, \ldots, f_N)$ slightly so that the mapping $(f_1/f_N, \ldots, f_{N-1}/f_N) : D_j \setminus \{f_N = 0\} \to \mathbf{C}^{N-1}$ is locally finite. Then we choose a constant $r > 1$ and an integer ν sufficiently large such that the mapping $g = (g_1, \ldots, g_{N-1})$ on D_j, where $g_j = (rf_j)^\nu - (rf_N)^\nu$ defines two analytic polyhedra of type $N-1$ which respectively approximate $\hat{K}_D$ externally and D internally. After $N-n$ such constructions, one obtains two special analytic polyhedra (i.e. of type n) which verify the following theorem. If $N(\epsilon^2/2) = n$, there is nothing to do.

Theorem 3. *For any $\epsilon > 0$ sufficiently small (such that $D(-\epsilon)$ is connected), there exist two integers $j \geq 2$ and $p \geq 2$ and there exist n holomorphic functions $f_1, \dots, f_n$ in $\mathcal{O}(D_j) \cap \mathcal{E}_{p,2j}$ such that*

$$K \subset \hat{K}_D \subset \overline{D(-1+\epsilon)} \subset \tilde{P}(-1+\epsilon+\beta(\epsilon)) \subset D(-1+\epsilon+\epsilon^2)$$

$$\text{and } \overline{D(-\epsilon)} \subset \tilde{P}(-\epsilon+\beta(\epsilon)) \subset D(-\epsilon+\epsilon^2).$$

$\tilde{P}(-1+\epsilon+\beta(\epsilon))$ and $\tilde{P}(-\epsilon+\beta(\epsilon))$ are two special holomorphic polyhedra. $\tilde{P}(-1+\epsilon+\beta(\epsilon))$ is the finite union of the connected components of the open set

$$\left\{ z \in D : \sup_{1 \leq l \leq n} \frac{1}{p} \log \mid f_l(z) \mid < -1+\epsilon+\beta(\epsilon) \right\}$$

that meet $\overline{D(-1+\epsilon)}$, and $\tilde{P}(-\epsilon+\beta(\epsilon))$ is the connected component containing $\overline{D(-\epsilon)}$, of the open set

$$\left\{ z \in D : \sup_{1 \leq l \leq n} \frac{1}{p} \log \mid f_l(z) \mid < -\epsilon+\beta(\epsilon) \right\},$$

where $0 < \beta(\epsilon) \leq \epsilon^2/2$.

Remark. If $N = N(\epsilon^2/2)$ in Corollary 2, we can choose $\beta(\epsilon) = \epsilon^2/2^{3(N-n)+1}$.

Third step. Pluricomplex Green functions with isolated logarithmic singularities. Let F be the holomorphic mapping given in Theorem 3 and defined by

$$F = (f_1, \dots, f_n) : \tilde{P}(-\epsilon+\beta(\epsilon)) \longrightarrow \mathbf{C}^n.$$

let $r_1 = \exp[p(-1+\epsilon+\beta(\epsilon))]$ and $r_0 = \exp[p(-\epsilon+\beta(\epsilon))]$. We denote by P_1 (resp. P_0) the polydisc in $\mathbf{C}^n$ centered in O with multiradius $r_1.(1, \dots, 1)$ (resp. $r_0.(1, \dots, 1)$). We prove that this mapping F is proper and surjective from the bounded special holomorphic polyhedron $\tilde{P}(-1+\epsilon+\beta(\epsilon))$ (respectively $\tilde{P}(-\epsilon+\beta(\epsilon))$) onto the polydisc P_1 (respectively P_0) (see part 3). Then F has a finite number of zeros in $\tilde{P}(-\epsilon+\beta(\epsilon))$, and has no zero on $\partial\tilde{P}(-\epsilon+\beta(\epsilon))$ nor on $\partial\tilde{P}(-1+\epsilon+\beta(\epsilon))$. Denote by $Z = \{p_1, \dots, p_k\}$ the finite set of zeros of F in the closure of $\tilde{P}(-\epsilon+\beta(\epsilon))$. We can suppose in addition that these points are ordered such that the first k' (always ≥ 1) zeros are exactly the zeros of F in $\tilde{P}(-1+\epsilon+\beta(\epsilon))$. We denote $Z' = \{p_j : 1 \leq j \leq k'\}$ (where $k' \leq k$) the finite set of zeros of F in $\tilde{P}(-1+\epsilon+\beta(\epsilon))$.

Define v to be the following Hartogs function on D_j:

$$v(z) = \sup_{1 \leq l \leq n} \left\{ \frac{\log(|f_l(z)|/r_0)}{\log(r_0/r_1)} \right\} = \sup_{1 \leq l \leq n} \frac{1}{1-2\epsilon} \left\{ \frac{1}{p} \log|f_l(z)| + \epsilon - \beta(\epsilon) \right\} \tag{1.7}$$

We remark that this function v belongs to a class of plurisubharmonic functions with isolated logarithmic poles. It is the object of part 4 to study these kinds of functions.

We can introduce the class of pluricomplex Green functions with isolated logarithmic poles p of growth $\log ||f||$ and of weight c ($c > 0$). Let D be an open subset of $\mathbf{C}^n$. Denote by P the finite set $\{(p_j, f_j, c_j), 1 \leq j \leq k\}$ where p_j are distinct poles in D, f_j are germs of holomorphic mappings in p_j and c_j are positive weights. We suppose that p_j is an isolated zero of the holomorphic mapping f_j around p_j, respectively for all j. We introduce the following extremal function:

$$\begin{aligned} g_D(P, z) = \sup\{u(z) : u \in PSH(D, [-\infty, 0)),\ \text{for } j = 1, \dots, k \\ u(z) - c_j \log ||f_j(z)|| \leq \text{Ø}(1) \text{ as } z \to p_j\}, \end{aligned} \tag{1.8}$$

where $PSH(D, [-\infty, 0))$ is the set of all negative plurisubharmonic (psh) functions on D. This function $g_D(P, .)$ is called the pluricomplex Green function of D with poles in P. We denote $\mathrm{mult}(f_j, p_j)$ the multiplicity of f_j at p_j, for $j = 1, \dots, k$. Now consider the following Dirichlet problem:

$$\begin{cases} u \in PSH(D) \cap \mathcal{C}(\overline{D}, [-\infty, 0]), \\ (dd^c u)^n = 0 \text{ on } D \setminus \{p_1, \dots, p_k\}, \\ u(z) - c_j \log ||f_j(z)|| = \text{Ø}(1) \text{ as } z \to p_j,\ \text{ for } 1 \leq j \leq k, \\ u(z) \to 0 \text{ as } z \to \partial D. \end{cases} \tag{1.9}$$

Theorem 4. *[LR99] If D is a bounded hyperconvex domain in $\mathbf{C}^n$ and if P is a finite set $\{(p_j, f_j, c_j), 1 \leq j \leq k\}$, of poles p_j in D associated respectively with the germs of the holomorphic mappings f_j and the positive weights c_j, then the function $u(z) = g_D(P, z)$ is the unique solution to the problem* (1.9).

In addition, it satisfies $(dd^c u)^n = (2\pi)^n \sum_{j=1}^{k} c_j^n \mathit{mult}(f_j, p_j)\delta_{p_j}$.

Proposition 5. *The function v, defined previously in* (1.7)*, is the pluricomplex Green function $g_{\tilde{P}(-\epsilon+\beta(\epsilon))}(P, .)$ in $\tilde{P}(-\epsilon + \beta(\epsilon))$, where P is the finite set $\{(p_j, F, \frac{1}{\log(r_0/r_1)}), 1 \leq j \leq k\}$, of poles $p_j \in Z$ associated respectively with the germs of the holomorphic map F and the weights $\frac{1}{\log(r_0/r_1)}$.*

Fourth step. Zahariuta's conjecture when D is strictly hyperconvex (part 5). v is identically equal to -1 on $\partial\tilde{P}(-1+\epsilon+\beta(\epsilon))$ and is identically equal to 0 on $\partial\tilde{P}(-\epsilon + \beta(\epsilon))$. But v has too many zeros in $\tilde{P}(-\epsilon + \beta(\epsilon))$. So we replace v by v', the pluricomplex Green function on $\tilde{P}(-\epsilon + \beta(\epsilon))$ with pole-set $P' = \{(p_j, F, \frac{1}{\log(r_0/r_1)}), 1 \leq j \leq k'\}$ in $\tilde{P}(-1 + \epsilon + \beta(\epsilon))$, associated respectively with the germs of the holomorphic map F and the weights $\frac{1}{\log(r_0/r_1)}$.

Then we approximate v' uniformly outside of $\tilde{P}(-1+\epsilon+\beta(\epsilon))$ (v' depends on ϵ) by a classical pluricomplex Green function g_ϵ on D. This can be done according to a result proved in part 4.

Then we use the Comparison Principle for the Monge-Ampère operator to prove that the family $\{g_\epsilon\}_\epsilon$ satisfies $\int_{\tilde{P}(-1+\epsilon+\beta(\epsilon))}(dd^c g_\epsilon)^n \to C(K, D)$ *when* ϵ *tends to 0.*

By applying a result of Poletsky and Nivoche [NP01] (see at the end of this paper for a detailed proof) which gives a sufficient condition for a sequence of pluricomplex Green functions to converge uniformly on any compact subset of $D \setminus K$ to the relative extremal function $u_{K,D}$, we conclude that g_ϵ is a uniform approximation of $u_{K,D}$ on any compact subset of $\overline{D} \setminus \hat{K}_D$.

So Theorem A is proved when D is strictly hyperconvex and it is easy to deduce Theorem B from this result.

Fifth and last step. Generalization to the hyperconvex case. If K is regular and D is bounded and hyperconvex, $u_{K,D} = u$ is a continuous exhaustion function for D. For any $\delta > 0$ sufficiently small, $D(-\delta) = \{z \in D : u(z) < -\delta\}$ is strictly hyperconvex and K is regular in $D(-\delta)$. By applying Theorem A to the couple $(K, D(-\delta))$ and according to the fact that $u_{K,D(-\delta)}$ converges uniformly to u on any compact set of D, we deduce Theorem A for the couple (K, D).

In the case where K is not necessarily regular in D, for any $\delta > 0$ sufficiently small, let K^δ denote the compact set of D defined by $K^\delta = \{z \in D : dist(z, \hat{K}_D) \leq \delta\}$.

Proposition 6. *Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ containing a compact set K. There exists a pluripolar set S in D such that $u_{K,D}$ is continuous on $\bar{D} \setminus S$. For any $\beta > 0$, $\epsilon > 0$ and $\delta > 0$ sufficiently small, there exists an open neighbourhood ω of S in D such that $C(\omega, D) < \beta$ and there exists g a classical pluricomplex Green function on D with a finite number of logarithmic poles such that*

(i) the poles of g are in the open neighbourhood $(K^\delta)^\circ$ of $\hat{K}_D$,
(ii) g satisfies the following uniform estimates on $\overline{D} \setminus ((K^\delta)^\circ \cup \omega)$

$$(1+\epsilon)g(z) \leq u_{K,D}(z) \leq (1-\epsilon)g(z).$$

2. Lelong and Bremermann's theorem for the relative extremal function

The proof of Theorem 1 is very similar to the proof of the corresponding result for the pluricomplex Green function with one logarithmic pole (see [Niv95]).

First, we give some preliminary properties of the functions h_p defined in (1.6).

Lemma 2.1. *Let D be a domain in $\mathbf{C}^n$ containing a compact set K. Then:*

(i) $\forall p \geq 1$, h_p *is a continuous psh function on D with values in* $]-\infty, 0[$.
(ii) $h_p(z) \leq -1$ *when* $z \in K$.
(iii) $(p+q)h_{p+q} \geq ph_p + qh_q$ *on D,* $\forall (p,q) \in (\mathbf{N}^*)^2$, *and the sequence* $(h_p)_{p\geq 1}$ *converges pointwise on D to* $\sup_{p\geq 1} h_p$.

Proof. For every $f \in \mathcal{E}_p$, $\frac{1}{p}\log \mid f \mid$ is a negative and continuous function on D, so h_p is lower semicontinuous on D. By using Montel's theorem, it follows that h_p is also upper semicontinuous on D. The plurisubharmonicity of h_p and property (ii) are a consequence of its definition.

If $f \in \mathcal{E}_p$ and $g \in \mathcal{E}_q$, $fg \in \mathcal{E}_{p+q}$. Consequently, $(p+q)h_{p+q} \geq \sup\{\log \mid fg \mid : f \in \mathcal{E}_p, g \in \mathcal{E}_q\} = ph_p + qh_q$. Then we deduce the pointwise convergence on D of the sequence $(h_p)_p$. □

Remark 2.2. From Lemma 2.1, we deduce that $h_{\alpha p} \geq h_p$, on D, $\forall (p, \alpha) \in (\mathbf{N}^*)^2$.

From the definition of $u_{K,D}$, we deduce the following lemma.

Lemma 2.3. *If D is a domain in $\mathbf{C}^n$ containing a compact set K, then for every $p \in \mathbf{N}^*$:*

$$h_p \leq u_{K,D}, \text{ on } D. \tag{2.10}$$

Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ containing a regular compact set K so that $u_{K,D}$ is continuous on $\overline{D}$. Demailly has introduced in [Dem92], for any $p \in \mathbf{N}^*$, the following space

$$\mathcal{H}_p = \left\{ f \in \mathcal{O}(D) : \int_D \mid f(z) \mid^2 e^{-2pu_{K,D}(z)} dV(z) < \infty \right\},$$

and the function u_p defined on D by

$$u_p(z) = \frac{1}{2p} \log \left(\sum_l \mid \sigma_l(z) \mid^2 \right).$$

$\mathcal{H}_p$ is an Hilbert space provided with the scalar product $(f,g)_p = \int_D f\overline{g}e^{-2pu_{K,D}}dV$ (the norm is denoted by $\|\ .\ \|_p$). We denote by dV the ordinary Lebesgue measure, and (σ_l) an orthonormal basis of $\mathcal{H}_p$.

Lemma 2.4. *[Dem92] For all $p \in \mathbf{N}^*$, we have* $u_p(z) = \sup_{f\in B_p} \frac{1}{p}\log|f(z)|$ *on D, where B_p is the unit ball in $\mathcal{H}_p$. Moreover, u_p is a continuous psh function on D.*

Demailly applied the Ohsawa-Takegoshi L^2-extension theorem ([Ohs88], Corollary 2) to prove the following theorem.

Theorem 2.5. *[Dem92] There exist a constant $c_1 > 0$, depending only on n and the diameter of D and a constant $c_2 > 0$, such that:*

$$u_p(z) \geq u_{K,D}(z) - \frac{c_1}{p}$$

and

$$u_p(z) \leq \sup_{w \in B(z,r)} u_{K,D}(w) + \frac{1}{p}(\log c_2 - n \log r)$$

for all z in D and all real numbers $r > 0$ such that $r < dist(z, bD)$. In particular, the sequence $(u_p)_{p \geq 1}$ converges pointwise to $u_{K,D}$ on D.

Now we suppose that D is a strictly hyperconvex domain in $\mathbf{C}^n$ containing a regular compact set K and D_j is the bounded hyperconvex domain defined by (1.4), for any integer $j \geq 1$.

Lemma 2.6. *Let D be a strictly hyperconvex domain in $\mathbf{C}^n$ containing a regular compact subset K. Then K is regular for D_j for any $j \geq 1$ and the sequence $(u_{K,D_j})_{j \geq 1}$ converges uniformly on $\overline{D}$ to $u_{K,D}$.*

Proof. Since $D \subset D_j$, then $u_{K,D} \geq u_{K,D_j} \geq -1$ and $u^*_{K,D} \geq u^*_{K,D_j} \geq -1$ on D. It is well known that K is regular for D if and only if $u^*_{K,D} \equiv -1$ on K (see [Kli91], p. 159). Thus $u^*_{K,D_j} \equiv -1$ on K, and K is also regular for D_j for any $j \geq 1$.

There exists $c > 0$ sufficiently large such that $c\varrho \leq -1$ on K and $c(\varrho - 1/j) \leq u_{K,D_j}$ on D_j. Then, $c_j = \inf_{z \in bD} u_{K,D_j}(z) \in [-1/j, 0)$ and $\lim_j c_j = 0$. For every $j \geq 1$, consider u_j the function defined on D_j by

$$u_j(z) = \begin{cases} u_{K,D_j}(z) & \text{if } z \in D_j \setminus D, \\ \max\{\, u_{K,D_j}(z),\ u_{K,D}(z) + c_j \,\} & \text{if } z \in \overline{D}. \end{cases}$$

u_j is a negative continuous psh function on D_j and $u_j \leq -1$ on K. Consequently $u_j \leq u_{K,D_j}$ on D_j and in particular, $u_{K,D} + c_j \leq u_{K,D_j}$ on $\overline{D}$. Thus we get the following statement : $u_{K,D_j} \leq u_{K,D} \leq u_{K,D_j} - c_j$ on $\overline{D}$, and the lemma is proved. □

Lemma 2.7. *We have the following inequality: $\forall j \geq 1$ and $\forall p \geq 1$*

$$\sup\left\{\frac{1}{p}\log|f| : f \in B_{p,j}\right\} = u_{p,j} \leq \frac{1}{p}\log(c_2\delta(j)^{-n}) + (1-\alpha_j)h_p \text{ on } D, \tag{2.11}$$

where $B_{p,j} = \{f \in \mathcal{O}(D_j) : \int_{D_j} |f|^2 e^{-2pu_{K,D_j}} dV \leq 1\}$, $\delta(j) = \mathrm{dist}(\overline{D}, bD_j)$ and α_j is a positive constant depending only on K, D and D_j which tends to 0 when j tends to ∞.

Proof. Let $f \in B_{p,j}$. There exists $z \in bD$ such that $| f(z) |= \sup\{| f(w) |: w \in \overline{D}\}$. By the Mean Value Inequality applied to the psh function $| f |^2$ on the ball $B(z, \delta(j)) \subset D_j$, we get:

$$| f(z) |^2 \leq \frac{c_2^2}{\delta(j)^{2n}} \int_{B(z,\delta(j))} | f(w) |^2 \, dV(w).$$

As u_{K,D_j} is negative on D_j, we deduce that

$$| f(z) |^2 \leq \frac{c_2^2}{\delta(j)^{2n}} \int_{B(z,\delta(j))} | f(w) |^2 \, e^{-2pu_{K,D_j}(w)} dV(w) \leq \frac{c_2^2}{\delta(j)^{2n}} \, || \, f \, ||_{p,j}^2,$$

where $|| \, f \, ||_{p,j}^2 = \int_{D_j} | f |^2 \, e^{-2pu_{K,D_j}} dV$. Also by the Mean Value Inequality applied to $| f |^2$, we obtain for any $z \in K$:

$$\begin{aligned} | f(z) |^2 &\leq \frac{c_2^2}{\delta(j)^{2n}} e^{-2p(1-\alpha_j)} \int_{B(z,\delta(j))} | f(w) |^2 \, e^{-2pu_{K,D_j}(w)} dV(w) \\ &\leq \frac{c_2^2}{\delta(j)^{2n}} e^{-2p(1-\alpha_j)} \, || \, f \, ||_{p,j}^2, \end{aligned}$$

where $-1 + \alpha_j = \sup_{z \in K_j} u_{K,D_j}(z)$ and $K_j = \{z \in D : \operatorname{dist}(z, K) \leq \delta(j)\}$. According to Lemma 2.6, α_j is positive and tends to 0 when j tends to ∞.

From Lemma 2.4 and the above inequalities, we obtain the required inequality (2.11). □

Finally, letting p, then j go to infinity in inequalities (2.10) and (2.11) and by Lemma 2.6 and Theorem 2.5, we derive that the sequence $(h_p)_{p \geq 1}$ converges pointwise to $u_{K,D}$ on D. This completes the proof of Theorem 1. □

3. Special holomorphic polyhedra and proper mappings

3.1. Holomorphic polyhedra

Let D be a strictly hyperconvex domain in $\mathbf{C}^n$ containing a regular compact set K. We obtain Corollary 2 directly from Theorem 1.

Proof of Corollary 2. First, according to Lemma 2.6, for any $\epsilon > 0$, there exists an integer $j \geq 1$ such that for all $j' \geq j$, we have

$$u_{K,D}(z) - \epsilon/3 \leq u_{K,D_{j'}}(z) \leq u_{K,D}(z) \text{ on } \overline{D}.$$

In addition, according to Theorem 1, we know that for any $j \geq 2$, $u_{K,D_j}(z) = \sup_{p \geq 1} h_{p,j}(z) = \lim_p h_{p,j}(z)$ pointwise in D_j, where $h_{p,j} = \sup\{\frac{1}{p} \log | f |: f \in \mathcal{E}_{p,j}\}$. And according to Remark 2.2, we have in particular for any $p \geq 2$

that the increasing sequence $(h_{p^\alpha,j})_\alpha$ converges pointwise to u_{K,D_j} on D_j. Consequently, by Dini's theorem, this sequence of continuous functions on D_j converges uniformly on any compact set of D_j to the continuous function u_{K,D_j}. If we fix any real number $\epsilon > 0$, any integer $p \geq 2$, and if we take the corresponding integer $j(\epsilon) \geq 1$ such that the first property is satisfied, then there exists an integer $\alpha_0 \geq 1$ such that for any $\alpha \geq \alpha_0$, we obtain on the compact set $\overline{D_{2j}}$ of D_j:

$$u_{K,D_j}(z) - \epsilon/3 \leq h_{p^\alpha,j}(z) \leq u_{K,D_j}(z).$$

Now for this integer α_0, there exist another integer $N \geq 1$ and N holomorphic functions $f_1, \dots, f_N \in \mathcal{E}_{p^{\alpha_0},j}$ such that

$$h_{p^{\alpha_0},j}(z) - \epsilon/3 \leq \sup_{1\leq l\leq N} \frac{1}{p^{\alpha_0}} \log | f_l(z) | \leq h_{p^{\alpha_0},j}(z) \text{ on } \overline{D_{2j}}.$$

The proof is complete. □

In what follows, to simplify the notation, we denote by u and u_j the relative extremal functions $u_{K,D}$ and u_{K,D_j} respectively.

Lemma 3.1. *Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ and let K be a compact set in D. Then for any $-1 < r < 0$ sufficiently near 0, the open set $D(r)$ (see (1.3)) is again connected.*

Proof. Choose any compact connected subset L of D with $K \subset L$. Denote $r_0 = \sup_{z\in L} u(z)$, $r_0 \in [-1, 0[$. Let $r_0 < r \leq 0$. Suppose that $D(r)$ is not connected. Denote $D(r, 1)$ the connected component of $D(r)$ which contains L. $D(r)$ has at least one other connected component, denoted by $D(r, 2)$. Let $\tilde{u}$ be the following negative psh function on D, defined by $\max\{u, r\}$ on $D(r, 2)$ and by u on $D \setminus D(r, 2)$. On $K \subset L \subset D(r, 1) \subset D \setminus D(r, 2)$, $\tilde{u} = u \equiv -1$. So by the definition of u, $\tilde{u} \leq u$ on D. This is a contradiction, since on $D(r, 2)$, $\tilde{u} = r > u$, and consequently the lemma is proved. □

If we fix ϵ, j, p, N and the N holomorphic functions $f_1, \dots, f_N$ as in Corollary 2, we denote by v_N the continuous psh function on D_j defined by

$$v_N(z) = \sup_{1\leq l\leq N} \frac{1}{p} \log | f_l(z) | .$$

For any $r \in IR$, we denote

$$P_N(r) = \{z \in D : v_N(z) < r\}.$$

We have the following inclusions $D(-\delta - \epsilon) \subset P_N(-\delta - \epsilon) \subset D(-\delta)$ for any $\delta > 0$, and in particular for $\epsilon^2/2$, $\delta = 1 - \epsilon - \epsilon^2$ and $\delta = \epsilon - \epsilon^2$, we obtain

$$D(-1 + \epsilon + \epsilon^2/2) \subset P_N(-1 + \epsilon + \epsilon^2/2) \subset D(-1 + \epsilon + \epsilon^2)$$
$$\text{and } D(-\epsilon + \epsilon^2/2) \subset P_N(-\epsilon + \epsilon^2/2) \subset D(-\epsilon + \epsilon^2).$$

3.2. Special holomorphic polyhedra

We use the same hypothesis as in the previous section. To prove the different results of Sect. 3.2, we are inspired by the proof of Bishop's Lemma about special holomorphic polyhedra. We refer to [Bis61], the original paper about this subject (see also [Nar60]). We recall that a *holomorphic polyhedron* of type N in an open set D in $\mathbf{C}^n$ is a finite union P of relatively compact connected components of a subset $\tilde{P} \subseteq D$ of the form $\tilde{P} = \{z \in D : |f_j(z)| < 1 \text{ for } j = 1, \dots, N\}$, where $f_j \in \mathcal{O}(D)$.

A holomorphic polyhedron of type n in a holomorphically convex open set D in $\mathbf{C}^n$ is called a *special holomorphic polyhedron.*

The existence of special holomorphic polyhedra is a rather nontrivial matter, and the principal existence result is the following ([Bis61], [Nar60]):

Bishop's lemma. *Suppose that D is a holomorphically convex open set in $\mathbf{C}^n$. Then whenever K is a holomorphically convex compact subset of D and U is an open neighbourhood of K in D, there is a special holomorphic polyhedron P such that $K \subset P \subseteq U$.*

Then Theorem 3 allows us to approximate $\hat{K}_D$ externally and D internally simultaneously by two special holomorphic polyhedra defined by the same n holomorphic functions.

If $N(\epsilon^2/2) = n$ in Corollary 2, we can easily deduce Theorem 3. Indeed, according to the notation at the end of Sect. 3.1, we have

$$\hat{K}_D \subset \overline{D(-1+\epsilon)} \subset D(-1+\epsilon+\epsilon^2/2) \subset \tilde{P}(-1+\epsilon+\epsilon^2/2) \subset D(-1+\epsilon+\epsilon^2),$$

where $\tilde{P}(-1+\epsilon+\epsilon^2/2)$ is the finite union of the connected components of the open set $P_n(-1+\epsilon+\epsilon^2/2)$ that meet $\overline{D(-1+\epsilon)}$. We also have

$$\overline{D(-\epsilon)} \subset D(-\epsilon+\epsilon^2/2) \subset \tilde{P}(-\epsilon+\epsilon^2/2) \subset D(-\epsilon+\epsilon^2),$$

where $\tilde{P}(-\epsilon+\epsilon^2/2)$ is the connected component of the open set $P_n(-\epsilon+\epsilon^2/2)$ that contains $\overline{D(-\epsilon)}$.

Now, if $N = N(\epsilon^2/2) > n$ in Corollary 2, the proof of Theorem 3 proceeds by induction by applying the processes $(\mathcal{P}_q)$ and $(\mathcal{R}_q)$ defined below successively, for $q = N, N-1, \dots, n+1$ respectively, until we obtain two holomorphic polyhedra of type n which are good exhaustions for $\hat{K}_D$ and D.

Until the end of Sect. 3.2, we suppose that the integer $N = N(\epsilon^2/2)$ given in Corollary 2 is greater than n, and we denote $p_N = p$.

3.2.1. Process $(\mathcal{P}_q)$, for $n+1 \leq q \leq N$

Proposition 3.2. *Let Ω be a domain in $\mathbf{C}^n$ containing a compact subset L. Let $f_1, \dots, f_q$ $(q > n)$ be q holomorphic functions in Ω. We suppose that f_q is not identically zero on Ω so that $X = \{z \in \Omega : f_q(z) = 0\}$ is a proper* *holomorphic subvariety in Ω. Then for any $\epsilon > 0$, there exist q holomorphic functions $\tilde{f}_1, \dots, \tilde{f}_q \in \mathcal{O}(\Omega)$ such that*

(i) $\sup_{1\le l\le q} \mid f_l(z) \mid e^{-\epsilon} \le \sup_{1\le l\le q} \mid \tilde{f}_l(z) \mid \le \sup_{1\le l\le q} \mid f_l(z) \mid, \forall z \in L,$

(ii) the holomorphic mapping $\left(\frac{\tilde{f}_1}{\tilde{f}_q}, \dots, \frac{\tilde{f}_{q-1}}{\tilde{f}_q}\right) : \Omega \setminus X \longrightarrow \mathbf{C}^{q-1}$ *is locally finite; that is, its level sets consist of just isolated points of* $\Omega \setminus X$.

Proof. Let $\epsilon > 0$ be fixed. We are going to use the following result, verified when $q > n$ ([Bis61], Theorem 1 and Lemma 4): *The set of elements $g = (g_1, \dots, g_{q-1})$ of $\mathcal{O}(\Omega)^{q-1}$ for which the mapping $(g_1 + f_1/f_q, \dots, g_{q-1} + f_{q-1}/f_q)$ is locally finite on $\Omega \setminus X$, is a dense subset of $\mathcal{O}(\Omega)^{q-1}$.*

Then for any $\epsilon' > 0$, there are functions $g_1, \dots, g_{q-1} \in \mathcal{O}(\Omega)$ such that $|| g_l ||_L \le \epsilon'$ and for which the mapping

$$\left(\frac{f_1}{f_q} + g_1, \dots, \frac{f_{q-1}}{f_q} + g_{q-1}\right) : \Omega \setminus X \longrightarrow \mathbf{C}^{q-1} \qquad (*)$$

is locally finite. It is at this point that the hypothesis $q > n$ is invoked.

If we apply this for $\epsilon' > 0$ such that $e^{-\epsilon/2} \le 1 - \epsilon' < 1 < 1 + \epsilon' \le e^{\epsilon/2}$ and if we denote

$$\tilde{f}_l = (f_l + g_l f_q)e^{-\epsilon/2} \text{ for } l = 1, \dots, q-1 \text{ and } \tilde{f}_q = f_q,$$

then we obtain that

$$\sup_{1\le l\le q} \mid \tilde{f}_l(z) \mid \le \sup_{1\le l\le q} \mid f_l(z) \mid, \forall z \in L.$$

Conversely, for any $z \in L$, there exists $l_0 \in \{1, \dots, q\}$ such that $\sup_{1\le l\le q} \mid f_l(z) \mid = \mid f_{l_0}(z) \mid$. If $l_0 = q$, then

$$\sup_{1\le l\le q} \mid \tilde{f}_l(z) \mid \ge \mid \tilde{f}_q(z) \mid = \sup_{1\le l\le q} \mid f_l(z) \mid .$$

If $1 \le l_0 \le q-1$, then $\mid \tilde{f}_{l_0}(z) \mid \ge (\mid f_{l_0}(z) \mid - \mid g_{l_0}(z) f_q(z) \mid) e^{-\epsilon/2}$, where $\mid g_{l_0}(z) \mid \le \epsilon'$ and $\mid f_q(z) \mid \le \mid f_{l_0}(z) \mid$. So

$$\mid \tilde{f}_{l_0}(z) \mid \ge \mid f_{l_0}(z) \mid (1-\epsilon') e^{-\epsilon/2} \ge \mid f_{l_0}(z) \mid e^{-\epsilon}.$$

Consequently, we obtain that

$$\sup_{1\le l\le q} \mid \tilde{f}_l(z) \mid \ge \sup_{1\le l\le q} \mid f_l(z) \mid e^{-\epsilon} \text{ on } L,$$

and (i) is proved. Property (ii) is a direct consequence of $(*)$. The proof is thereby complete. □

Process $(\mathcal{P}_q)$. Suppose that we have q holomorphic functions $f_l \in \mathcal{E}_{p_q,2j} \cap \mathcal{O}(D_j)$ such that the function f_q is not identically zero on the domain D_j, so that $X_{j,q} = \{z \in D_j : f_q(z) = 0\}$ is a proper holomorphic subvariety in D_j. We denote by v_q the continuous psh function given by

$$v_q = \sup_{1 \le l \le q} \frac{1}{p_q} \log \mid f_l \mid,$$

and for any $r \in IR$,

$$P_q(r) = \{z \in D : v_q(z) < r\}.$$

Suppose also that the following inclusions are satisfied:

$$\overline{D(-1+\epsilon)} \subset P_q(-1+\epsilon+\epsilon^2/2^{1+3(N-q)}, 1) \subset D(-1+\epsilon+\epsilon^2), \quad (3.12)$$

$$\overline{D(-\epsilon)} \subset P_q(-\epsilon+\epsilon^2/2^{1+3(N-q)}, 1) \subset D(-\epsilon+\epsilon^2), \quad (3.13)$$

where $P_q(-1+\epsilon+\epsilon^2/2^{1+3(N-q)}, 1)$ is the finite union of the connected components of the open set $P_q(-1+\epsilon+\epsilon^2/2^{1+3(N-q)})$ that meet $\overline{D(-1+\epsilon)}$, and $P_q(-\epsilon+\epsilon^2/2^{1+3(N-q)}, 1)$ is the connected component of the open set $P_q(-\epsilon+\epsilon^2/2^{1+3(N-q)})$ that contains $\overline{D(-\epsilon)}$.

Then we apply Proposition 3.2 to these q holomorphic functions f_l for $p_q\epsilon^2/2^{2+3(N-q)}$, $\Omega = D_j$ and $L = \overline{D_{2j}}$. We obtain q holomorphic functions $\tilde{f}_1, \ldots, \tilde{f}_q$ on D_j such that

(i) $v_q(z) - \epsilon^2/2^{2+3(N-q)} \le \sup_{1 \le l \le q} \frac{1}{p_q} \log \mid \tilde{f}_l(z) \mid \le v_q(z) \le u_{2j}(z)$ on $\overline{D_{2j}}$,
(ii) in particular, $f_l \in \mathcal{E}_{p_q,2j}$ for any $1 \le l \le q$,
(iii) the following holomorphic mapping is locally finite:

$$\left(\frac{\tilde{f}_1}{\tilde{f}_q}, \ldots, \frac{\tilde{f}_{q-1}}{\tilde{f}_q}\right) : D_j \setminus X_{j,q} \longrightarrow \mathbf{C}^{q-1}. \quad (3.14)$$

We denote by $\tilde{v}_q$ the continuous psh function given by $\tilde{v}_q = \sup_{1 \le l \le q} \frac{1}{p_q} \log \mid \tilde{f}_l \mid$, and for any $r \in IR$,

$$\tilde{P}_q(r) = \{z \in D : \tilde{v}_q(z) < r\}.$$

Then we obtain the following inclusions for any $\delta > 0$,

$$D\left(-\delta - \frac{\epsilon^2}{2^{2+3(N-q)}}\right) \subset P_q\left(-\delta - \frac{\epsilon^2}{2^{2+3(N-q)}}\right) \subset \tilde{P}_q\left(-\delta - \frac{\epsilon^2}{2^{2+3(N-q)}}\right) \subset P_q(-\delta).$$

In particular, for $\delta = 1 - \epsilon - \epsilon^2/2^{1+3(N-q)}$ and $\delta = \epsilon - \epsilon^2/2^{1+3(N-q)}$, we have respectively

$$K \subset \hat{K}_D \subset \overline{D(-1+\epsilon)} \subset \tilde{P}_q(-1+\epsilon+\epsilon^2/2^{2+3(N-q)}, 1) \subset D(-1+\epsilon+\epsilon^2),$$

$$\overline{D(-\epsilon)} \subset \tilde{P}_q(-\epsilon+\epsilon^2/2^{2+3(N-q)}, 1) \subset D(-\epsilon+\epsilon^2),$$

where $\tilde{P}_q(-1+\epsilon+\epsilon^2/2^{2+3(N-q)}, 1)$ is the finite union of the connected components of the open set $\tilde{P}_q(-1+\epsilon+\epsilon^2/2^{2+3(N-q)})$ that meet $\overline{D(-1+\epsilon)}$, and $\tilde{P}_q(-\epsilon+\epsilon^2/2^{2+3(N-q)}, 1)$ is the connected component of the open set $\tilde{P}_q(-\epsilon+\epsilon^2/2^{2+3(N-q)})$ that contains $\overline{D(-\epsilon)}$.

3.2.2. Process $(\mathcal{R}_q)$ for $n+1 \leq q \leq N$. We use the same notations as at the end of Sect. 3.2.1. Let us denote, for any $1 \leq l \leq q-1$ and for any $\nu \in I\!N^*$, the holomorphic function on D_j

$$F_{l,\nu} = \tilde{f}_l^\nu - \tilde{f}_q^\nu.$$

For any $r \in I\!R$, we denote by $P^\nu_{q-1}(r)$ the open set in D defined by

$$P^\nu_{q-1}(r) = \left\{z \in D : w^\nu_{q-1}(z) < r\right\},$$

where $w^\nu_{q-1} = \sup_{1\leq l\leq q-1} \frac{1}{p_q\nu} \log \mid F_{l,\nu} \mid$ is a continuous psh function on D_j.

Proposition 3.3. *For any $\epsilon > 0$ sufficiently small (such that $D(-\epsilon)$ is connected), if $\alpha_1(\epsilon) = \epsilon^2/2^{2+3(N-q)}$ and $\alpha_2(\epsilon) = \epsilon^2/2^{3(N-q+1)}$, there exists an integer $\nu_q \geq 1$ such that for any $\nu \geq \nu_q$, we have*

$$\overline{D(-1+\epsilon)} \subset P^\nu_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1) \subset \tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon), 1), \quad (3.15)$$

where $P^\nu_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1)$ is the finite union of the connected components of $P^\nu_{q-1}(-1+\epsilon+\alpha_2(\epsilon))$ that meet $\overline{D(-1+\epsilon)}$, and

$$\overline{D(-\epsilon)} \subset P^\nu_{q-1}(-\epsilon+\alpha_2(\epsilon), 1) \subset \tilde{P}_q(-\epsilon+\alpha_1(\epsilon), 1), \quad (3.16)$$

where $P^\nu_{q-1}(-\epsilon+\alpha_2(\epsilon), 1)$ is the connected component of $P^\nu_{q-1}(-\epsilon+\alpha_2(\epsilon))$ that contains $\overline{D(-\epsilon)}$.

Proof. For any $l \in \{1, \ldots, q-1\}$ and for any $\nu \geq 1$, $F_{l,\nu}$ is a holomorphic function on D_j which satisfies $\mid F_{l,\nu}(z) \mid \leq 2 \sup_{1\leq l\leq q} \mid \tilde{f}_l(z) \mid^\nu$ on D_j. Consequently, $\| F_{l,\nu} \|_K \leq 2e^{-p_q\nu}$, $\| F_{l,\nu} \|_{\overline{D_{2j}}} \leq 2$, and

$$\begin{aligned}\sup_{1\leq l\leq q-1} \frac{1}{p_q\nu} \log \mid F_{l,\nu}(z) \mid &\leq \sup_{1\leq l\leq q} \frac{1}{p_q} \log \mid \tilde{f}_l(z) \mid + \frac{\log 2}{p_q\nu} \text{ on } D_j, \\ &\leq u(z) + \frac{\log 2}{p_q\nu} \text{ on } \overline{D}.\end{aligned}$$

We have

$$\hat{K}_D \subset \overline{D(-1+\epsilon)} \subset \tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon), 1) \subset D(-1+\epsilon+\epsilon^2).$$

If we choose a positive constant c_2 such that $0 < \alpha_2(\epsilon) < c_2 < \alpha_1(\epsilon)$ and an integer ν_q^0 such that $\log 2/(p_q\nu) < \alpha_2(\epsilon)$ for $\nu \geq \nu_q^0$, then we obtain that

$$\overline{D(-1+\epsilon)} \subset P^{\nu}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1).$$

To complete the proof of relation (3.15), it is enough to show that whenever ν is sufficiently large,

$$P^{\nu}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1) \subset \tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon)).$$

If it was not the case, then there would be a sequence $(\nu_k)_k$ of integers such that $\nu_k \to \infty$ and

$$P^{\nu_k}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1) \not\subset \tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon)).$$

In what follows, we will consider integers k sufficiently large (and the corresponding ν_k) such that

$$\hat{K}_D \subset \overline{D(-1+\epsilon)} \subset P^{\nu_k}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1).$$

For each connected component $P^{\nu_k,0}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1)$ of $P^{\nu_k}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1)$, necessarily $P^{\nu_k,0}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1) \cap \overline{D(-1+\epsilon)} \neq \emptyset$, and there must be some connected component $P^{\nu_k,0}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1)$ for which

$$P^{\nu_k,0}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1) \cap \partial\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon)) \neq \emptyset.$$

Now introduce the auxiliary open level set $\tilde{P}_q(-1+\epsilon+c_2)$ for which $\overline{\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))} \setminus \tilde{P}_q(-1+\epsilon+c_2)$ is compact in D. We note that there must be some connected component

$$R_k \subset P^{\nu_k,0}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1) \cap [\overline{\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))} \setminus \tilde{P}_q(-1+\epsilon+c_2)]$$

for which

$$R_k \cap \partial\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon)) \neq \emptyset \text{ and } R_k \cap \partial\tilde{P}_q(-1+\epsilon+c_2) \neq \emptyset.$$

Indeed, choose a path in $P^{\nu_k,0}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1)$ from a point in $P^{\nu_k,0}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1) \cap \overline{D(-1+\epsilon)}$ to a point in $P^{\nu_k,0}_{q-1}(-1+\epsilon+\alpha_2(\epsilon), 1) \cap \partial\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon)) \neq \emptyset$. Then observe that the segment of that path from the last point in $\overline{\tilde{P}_q(-1+\epsilon+c_2)}$ to the first point in $\partial\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))$ belongs to such a connected component R_k. If $z \in R_k$, then $z \in P^{\nu_k}_{q-1}(-1+\epsilon+\alpha_2(\epsilon))$,

$$|\, \tilde{f}_l(z)^{\nu_k} - \tilde{f}_q(z)^{\nu_k} \,| < e^{p_q\nu_k(-1+\epsilon+\alpha_2(\epsilon))}, \text{ for } l = 1, \ldots, q-1,$$

and since $z \notin \tilde{P}_q(-1+\epsilon+c_2)$ necessarily

$$| \tilde{f}_{l_0}(z) | \geq e^{p_q(-1+\epsilon+c_2)} \text{ for some index } 1 \leq l_0 \leq q.$$

Combining these last two inequalities, we obtain a third relation

$$\begin{aligned} | \tilde{f}_q(z) |^{\nu_k} &\geq | \tilde{f}_{l_0}(z) |^{\nu_k} - | \tilde{f}_q(z)^{\nu_k} - \tilde{f}_{l_0}(z)^{\nu_k} | \\ &> e^{p_q \nu_k(-1+\epsilon+c_2)} - e^{p_q \nu_k(-1+\epsilon+\alpha_2(\epsilon))} > 0 \end{aligned} \tag{3.17}$$

from which it follows that $R_k \subset D_j \setminus X_{j,q}$, and hence the mapping

$$\left(\frac{\tilde{f}_1}{\tilde{f}_q}, \ldots, \frac{\tilde{f}_{q-1}}{\tilde{f}_q} \right) : D_j \setminus X_{j,q} \longrightarrow \mathbf{C}^{q-1}$$

is well defined on R_k. Then, combining the above inequalities yields the following inequality:

$$\begin{aligned} \left| \frac{\tilde{f}_l(z)^{\nu_k}}{\tilde{f}_q(z)^{\nu_k}} - 1 \right| &= \left| \frac{\tilde{f}_l(z)^{\nu_k} - \tilde{f}_q(z)^{\nu_k}}{\tilde{f}_q(z)^{\nu_k}} \right| \\ &< \frac{e^{p_q \nu_k(-1+\epsilon+\alpha_2(\epsilon))}}{e^{p_q \nu_k(-1+\epsilon+c_2)} - e^{p_q \nu_k(-1+\epsilon+\alpha_2(\epsilon))}} = \frac{1}{e^{p_q \nu_k(c_2-\alpha_2(\epsilon))} - 1}, \end{aligned}$$

and since $e^{p_q(c_2-\alpha_2(\epsilon))} > 1$, it follows from this that

$$\left| \frac{\tilde{f}_l(z)^{\nu_k}}{\tilde{f}_q(z)^{\nu_k}} - 1 \right| < \frac{\pi}{\nu_k} \text{ for } \nu_k \text{ sufficiently large.}$$

Geometrically, this last inequality means that the point $\tilde{f}_l(z)/\tilde{f}_q(z)$ lies within one of ν_k disjoint open neighborhoods of the ν_k roots of unity, where these neighborhoods have the property that their radii tend to zero as ν_k increases to infinity.

Now, since R_k is connected, the points $\tilde{f}_l(z)/\tilde{f}_q(z)$ must indeed lie in the same neighborhood for all $z \in R_k$. The mapping $z \in R_k \longrightarrow \left(\frac{\tilde{f}_1(z)}{\tilde{f}_q(z)}, \ldots, \frac{\tilde{f}_{q-1}(z)}{\tilde{f}_q(z)} \right)$ thus takes R_k into a product of $q-1$ such neighborhoods. After passing to a suitable subsequence of the indices ν_k if necessary, it can be assumed that these neighborhoods shrink to a single point $(\xi_1, \ldots, \xi_{q-1})$, where of course $| \xi_l | = 1$. Thus for any points $z_k \in R_k$,

$$\lim_{k \to \infty} \frac{\tilde{f}_l(z_k)}{\tilde{f}_q(z_k)} = \xi_l, \ 1 \leq l \leq q-1.$$

Now for any value t in the interval $[c_2, \alpha_1(\epsilon)]$, there must be some point $z_t^k \in R_k \subset \overline{\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))} \setminus \tilde{P}_q(-1+\epsilon+c_2)$ for which $\sup_l |\tilde{f}_l(z_t^k)| =$

$e^{p_q(-1+\epsilon+t)}$, and since $\overline{\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))} \setminus \tilde{P}_q(-1+\epsilon+c_2) \subset \overline{\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))}$, which is a compact set in D, a subsequence of these points will converge to a limit point $z_t \in \overline{\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))} \setminus \tilde{P}_q(-1+\epsilon+c_2)$. Since $\sup_l |\tilde{f}_l(z_t)| = e^{p(-1+\epsilon+t)}$, these points are distinct for distinct values of t, so there are indeed uncountably many such points.

The values $\tilde{f}_q(z_t^k)$ are uniformly bounded away from zero as a consequence of (3.17). Indeed,

$$\begin{aligned}\left|\tilde{f}_q\left(z_t^k\right)\right| &\geq e^{p_q(-1+\epsilon+c_2)}(1-e^{p_q\nu_k(\alpha_2(\epsilon)-c_2)})^{1/\nu_k}\\ &= e^{p_q(-1+\epsilon+c_2)}\exp\left(\frac{1}{\nu_k}\log(1-e^{p_q\nu_k(\alpha_2(\epsilon)-c_2)})\right),\end{aligned}$$

where the last factor $\exp(\frac{1}{\nu_k}\log(1-e^{p_q\nu_k(\alpha_2(\epsilon)-c_2)}))$ tends to 1 when k tends to ∞. Thus $|\tilde{f}_q(z_t)| \neq 0$, and consequently $z_t \notin X_{j,q}$. It then follows that all the points z_t have the same image under the mapping (3.14), contradicting the condition that the mapping is locally finite and hence can have at most countably many inverse images. That contradiction means that it must be the case that for $\alpha_2(\epsilon)$ chosen as before, $P_{q-1}^{\nu}(-1+\epsilon+\alpha_2(\epsilon), 1) \subset \tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))$ whenever ν is sufficiently large (i.e. for $\nu \geq \nu_q^1 \geq \nu_q^0$) . In addition, each connected component of $P_{q-1}^{\nu}(-1+\epsilon+\alpha_2(\epsilon), 1)$ is contained in a connected component of $\tilde{P}_q(-1+\epsilon+\alpha_1(\epsilon))$, which then itself meets $\overline{D(-1+\epsilon)}$. Consequently, the inclusions (3.15) are proved. In the same way as for (3.15), we can prove that there exists an integer $\nu_q^2 \geq \nu_q^0$ such that (3.16) holds for any $\nu \geq \nu_q^2$. The proof of Proposition 3.3 is complete. $\square$

Process $(\mathcal{R}_q)$ first consists in applying Proposition 3.3 to the $q-1$ holomorphic functions $F_{l,\nu}$, defined at the beginning of Sect. 3.2.2. Then we choose ν sufficiently large such that $\log 2/(p_q\nu) \leq \epsilon^2/2^{3(N-q+1)+1}$, and we denote by $p_{q-1} = p_q\nu$. We define $q-1$ new holomorphic functions f_l by $F_{l,\nu}/2 \in \mathcal{E}_{p_{q-1},2j} \cap \mathcal{O}(D_j)$. We also denote by $P_{q-1}(r)$ the open sublevel set

$$P_{q-1}(r) = \{z \in D : v_{q-1}(z) < r\},$$

where v_{q-1} is the psh function defined by

$$v_{q-1} = \sup_{1\leq l\leq q-1} \frac{1}{p_{q-1}} \log |\ f_l\ | \ .$$

Remark that $v_{q-1} \leq u$ on D and that $P_{q-1}(r) = P_{q-1}^{\nu}(r+\log 2/p_{q-1})$ for any $r < 0$. Since the inclusions (3.15) and (3.16) are satisfied and ν is sufficiently large such that $\log 2/(p_q\nu) \leq \epsilon^2/2^{3(N-q+1)+1}$, we deduce that

$$\overline{D(-1+\epsilon)} \subset P_{q-1}(-1+\epsilon+\epsilon^2/2^{3(N-q+1)+1}, 1) \subset D(-1+\epsilon+\epsilon^2)$$

and
$$\overline{D(-\epsilon)} \subset P_{q-1}(-\epsilon + \epsilon^2/2^{3(N-q+1)+1}, 1) \subset D(-\epsilon + \epsilon^2),$$
where $P_{q-1}(-1+\epsilon+\epsilon^2/2^{3(N-q+1)+1}, 1)$ is the finite union of the connected components of $P_{q-1}(-1+\epsilon+\epsilon^2/2^{3(N-q+1)+1})$ that meet $\overline{D(-1+\epsilon)}$, and $P_{q-1}(-\epsilon+\epsilon^2/2^{3(N-q+1)+1}, 1)$ is the connected component of $P_{q-1}(-\epsilon + \epsilon^2/2^{3(N-q+1)+1})$ that contains $\overline{D(-\epsilon)}$.

3.2.3. Proof of Theorem 3: iteration of the processes ($\mathcal{P}_q$) and ($\mathcal{R}_q$) successively for $q = N, N-1, \ldots, n+1$ respectively. We use the same notations as at the end of Sect. 3.1 and as in Sects. 3.2.1 and 3.2.2 for $q = N = N(\epsilon^2/2)$ (in Corollary 2). According to Corollary 2, there exist N ($> n$) holomorphic functions $f_1, \ldots, f_N \in \mathcal{E}_{p_N,j} \subset \mathcal{E}_{p_N,2j} \cap \mathcal{O}(D_j)$ such that (3.12) and (3.13) are verified for $q = N$.

We can assume of course that the function f_N is not identically zero on the domain D_j so that $X_{j,N} = \{z \in D_j : f_N(z) = 0\}$ is a proper holomorphic subvariety in D_j.

Therefore, we are able to apply processes ($\mathcal{P}_N$) and ($\mathcal{R}_N$) in turn to obtain the existence of $N-1$ new holomorphic functions $f_l \in \mathcal{E}_{p_{N-1},2j} \cap \mathcal{O}(D_j)$ ($p_{N-1} \geq p_N$) such that (3.12) and (3.13) are verified for $q = N-1$.

So we have the necessary hypothesis to apply the processes ($\mathcal{P}_{N-1}$) and ($\mathcal{R}_{N-1}$) again. It is clear now that, by iterating processes ($\mathcal{P}_q$) and ($\mathcal{R}_q$), for $q = N-1$, for $q = N-2, \ldots$ and finally for $q = n+1$, we obtain the existence of n new holomorphic functions $f_l \in \mathcal{E}_{p_n,2j} \cap \mathcal{O}(D_j)$ ($p_n \geq p_N$) which satisfy the conclusion of Theorem 3. □

3.3. Proper mappings

In this section, D is once again a strictly hyperconvex domain in $\mathbf{C}^n$ containing a regular compact set K. We use the same notations as in Theorem 3. Let $r_1 = \exp[p(-1+\epsilon+\beta(\epsilon))]$ and $r_0 = \exp[p(-\epsilon+\beta(\epsilon))]$. We call P_1 (resp. P_0) the polydisc in $\mathbf{C}^n$ centered in O with multiradius $r_1.(1, \ldots, 1)$ (resp. $r_0.(1, \ldots, 1)$). Let F be the following holomorphic mapping defined by
$$F = (f_1, \ldots, f_n) : \tilde{P}(-\epsilon + \beta(\epsilon)) \longrightarrow \mathbf{C}^n.$$

Proposition 3.4. *The mapping F is proper and surjective from the bounded special holomorphic polyhedron $\tilde{P}(-\epsilon+\beta(\epsilon))$ (respectively $\tilde{P}(-1+\epsilon+\beta(\epsilon))$) to the polydisc P_0 (respectively P_1).*

Proof. To prove that F is proper from $\tilde{P}(-\epsilon+\beta(\epsilon))$ to P_0 (respectively from $\tilde{P}(-1+\epsilon+\beta(\epsilon))$ to P_1), we just verify that if $(z_k)_k$ is a sequence in $\tilde{P}(-\epsilon+\beta(\epsilon))$ (resp. $\tilde{P}(-1+\epsilon+\beta(\epsilon))$) which converges to a boundary point $z_0 \in \partial\tilde{P}(-\epsilon+\beta(\epsilon))$ (resp. $\partial\tilde{P}(-1+\epsilon+\beta(\epsilon))$), then the sequence $(F(z_k))_k$ converges to the boundary point $F(z_0) \in \partial P_0$ (resp. ∂P_1).

According to Remmert's proper mapping theorem ([Loj91], p. 290 and 300), since $F : \tilde{P}(-\epsilon+\beta(\epsilon)) \to P_0$ (respectively $\tilde{P}(-1+\epsilon+\beta(\epsilon)) \to P_1$) is proper and $\dim \tilde{P}(-\epsilon + \beta(\epsilon)) = \dim \tilde{P}(-1 + \epsilon + \beta(\epsilon)) = \dim P_0 = \dim P_1 = n$, F is surjective. □

4. Extremal functions with logarithmic singularities

The function v defined by (1.7) belongs to the class of pluricomplex Green functions with isolated logarithmic poles p of growth $\log ||f||$ and weight c ($c > 0$), where f is a holomorphic mapping from an open set of $\mathbf{C}^n$ to $\mathbf{C}^n$. In this part, we study some properties of this class. Some results are particular cases of results proved by Lelong and Rashkovskii in [LR99].

Let D be an open subset in $\mathbf{C}^n$ and let p be a point in D. Let $f^1, \ldots, f^n$ be n germs of holomorphic functions in p such that p is an isolated zero of the holomorphic mapping $f = (f^1, \ldots, f^n)$ around p. Let c be a positive real number. If u is a psh function in a neighbourhood of p, we will say that *u has a logarithmic pole at p of growth* $\log ||f||$ *and weight c* if

$$u(z) - c \log ||f(z)|| \leq \emptyset(1) \text{ as } z \to p.$$

In particular, if $f(z) = z - p$, we just say that *u has a logarithmic pole at p of weight c* ([Kli91]).

Let $p_1, \ldots, p_k$ be k distinct points in D. For each point p_j, let $f_j^1, \ldots, f_j^n$ be n germs of holomorphic functions in p_j such that p_j is an isolated zero of the holomorphic mapping $f_j = (f_j^1, \ldots, f_j^n)$ around p_j, and let $c_1, \ldots, c_k$ be k positive weights. Denote by P the finite set $\{(p_j, f_j, c_j), 1 \leq j \leq k\}$. As for the classical pluricomplex Green function, we define an extremal function $g_D(P, .)$ by (1.8) and we call it the *pluricomplex Green function on D with poles in P*.

Lemma 4.1. *Let p be a point in $\mathbf{C}^n$ and f be a germ of holomorphic map such that p is an isolated zero of f in V, an open neighbourhood of p. If $u(z) = \log ||f(z)||$ in V, then $(dd^c u)^n = (2\pi)^n mult(f, p)\delta_p$ on V, where $mult(f, p)$ is the algebraic or geometric multiplicity of f at p.*

Proof. Let us denote by $m = \text{mult}(f, p)$. $1 \leq m < +\infty$. If $m = 1$, then p is a classical logarithmic pole of weight 1 for u, and there exist two constants c_1 and c_2 such that $c_1 + \log ||z - p|| \leq u(z) \leq \log ||z - p|| + c_2$.

If $m > 1$, there exists a sequence $(v^\nu)_\nu$ of regular values for f converging to O. Denote by f^ν the holomorphic mapping defined on V by $f^\nu = f - v^\nu$, and let $(f^\nu)^{-1}(O) = \{w_1^\nu, \ldots, w_m^\nu\}$. For ν sufficiently large, all the points $w_1^\nu, \ldots, w_m^\nu$ are distinct and regular for f^ν and they converge to p when ν tends to infinity.

$(dd^c \log ||f^\nu||)^n = 0$ on $V \setminus (f^\nu)^{-1}(O)$ and when we take the limit, we obtain that $(dd^c \log ||f||)^n = 0$ on $V \setminus \{p\}$.

$(dd^c \log ||f^\nu||)^n(z) = (2\pi)^n \sum_{l=1}^m \delta_{w_l^\nu}(z)$ on V. For any $r > 0$ sufficiently small, there exists $\nu_0 \geq 1$ such that for any $\nu \geq \nu_0$ we have $\int_{B(p,r)} (dd^c \log ||f^\nu||)^n(z) = (2\pi)^n m$, and thus $\int_{B(p,r)} (dd^c \log ||f||)^n(z) = (2\pi)^n m$. Consequently, $(dd^c \log ||f||)^n(p) = (2\pi)^n m$. □

In the next proposition, we list some basic properties of the extremal function $g_D(P, .)$.

Proposition 4.2. *[LR99] Let D, D' be two open sets in $\mathbf{C}^n$ and let P be a finite set $\{(p_j, f_j, c_j), 1 \leq j \leq k\}$, of poles p_j in D associated respectively with the germs of the holomorphic mappings f_j and the weights c_j. Then the following statements hold:*

(i) If $z \in D$ and $D \subset D'$, then $g_D(P, z) \geq g_{D'}(P, z)$.
(ii) If $z \in D$, $D \subset D'$ and $D' \setminus D$ is pluripolar, then $g_D(P, z) = g_{D'}(P, z)$.
(iii) Suppose that D is bounded. Then there exists a constant $C \in IR$ such that for any $j = 1, \ldots, k$,

$$g_D(P, z) \geq c_j \log ||f_j(z)|| + C \text{ near } p_j. \tag{4.18}$$

If $r_j > 0$ and $\{z \in V_j : ||f_j(z)|| < r_j\} \Subset V_j \subset D$ for $j = 1, \ldots, k$ (V_j is a neighbourhood of p_j where the mapping f_j is defined, bounded by 1 and has p_j as unique zero), then

$$g_D(P, z) \leq c_j \log(||f_j(z)||/r_j) \text{ on } \{z \in V_j : ||f_j(z)|| < r_j\}. \tag{4.19}$$

(iv) If D is bounded, then $z \mapsto g_D(P, z)$ is a negative psh function with k logarithmic poles p_j of growth $\log ||f_j||$ and of weight c_j, respectively for $j = 1, \ldots, k$.
(v) If D is a bounded hyperconvex domain, then for any $w \in \partial D$, $\lim_{\substack{z \to w \\ z \in D}} g_D(P, z) = 0$.
(vi) If D is bounded, then $z \mapsto g_D(P, z)$ is maximal in $D \setminus \{p_1, \ldots, p_k\}$, i.e.

$$(dd^c g_D(P, .))^n \equiv 0 \text{ in } D \setminus \{p_1, \ldots, p_k\}.$$

Proposition 4.3. *Let D and D' be two bounded domains in $\mathbf{C}^n$ and $F : D \to D'$ be a holomorphic mapping.*

(i) Let P be the finite set $\{(p_j, f_j, c_j), 1 \leq j \leq k\}$ of poles p_j in D associated respectively with the germs of the holomorphic mappings f_j and the weights c_j. Let P' be the finite set $\{(F(p_j), g_j, c_j), 1 \leq j \leq k\}$ of poles $F(p_j)$ in D' associated respectively with the germs of the holomorphic maps g_j and the weights c_j. We suppose that $g_j \circ F = f_j$ and that $g_j = g_{j'}$ if $F(p_j) = F(p_{j'})$, then

$$g_{D'}(P', F(z)) \leq g_D(P, z), \ z \in D.$$

(ii) Let F be a proper holomorphic mapping and p' be a point in D' such that $F^{-1}(\{p'\})$ is the finite set of points in D $\{p_j : 1 \leq j \leq k\}$. Let P be

the finite set $\{(p_j, F - p'), 1 \leq j \leq k\}$ *of poles* p_j *in* D *associated with the germs of the holomorphic mapping* $F - p'$ *and weights equal to 1. Then*

$$g_D(P, z) = g_{D'}(p', F(z)) \text{ on } D.$$

Proof. $g_{D'}(P', F(.))$ is a negative psh function on D such that $g_{D'}(P', F(z)) - c_j \log ||f_j(z)|| \leq \emptyset(1)$, as $z \to p_j$, for $j = 1, \dots, k$. Then $g_{D'}(P', F(.)) \leq g_D(P, .)$ on D and the first item is proved.

Let us proved now the second item. Set $u = g_D(P, .)$ and define $v(w) = \sup\{u(z) : z \in F^{-1}(w)\}$, on D'. It is well known that the function v is psh and negative on D' (see [Kli91]). By construction, $u \leq v \circ F$ on D. Furthermore, v has a logarithmic pole in p'. Indeed, when w tends to p', $z \in F^{-1}(w)$ tends to some point in $\{p_1, \dots, p_k\}$, $u(z) \leq \log ||F(z) - p'|| + C$ around p_j, $1 \leq j \leq k$ (where C is a constant) and $v(w) \leq \log ||w - p'|| + C$. Consequently, we have the first inequality $g_D(P, .) \leq g_{D'}(p', F(.))$ on D. For the converse, apply item (i). □

Now Theorem 4 gives us alternative description of the pluricomplex Green function $g_D(P, .)$ in terms of the Monge-Ampère operator.

Let D be a bounded hyperconvex domain in $\mathbf{C}^n$ and let P be a finite set $\{(p_j, f_j, c_j), 1 \leq j \leq k\}$, of poles p_j in D associated respectively with the germs of the holomorphic mappings f_j and the positive weights c_j. Fix $r > 0$ sufficiently small such that f_j is defined in a neighbourhood of $\overline{B(p_j, r)}$ and such that p_j is the unique zero of f_j in $\overline{B(p_j, r)}$, for $j = 1, \dots, k$. The multiplicity of f_j at p_j is equal to the number of preimages in $\overline{B(p_j, r)}$ of any regular value for f_j sufficiently near O. Denote $m_j = \text{mult}(f_j, p_j)$; $m_j \geq 1$. If $m_j > 1$, there exists a sequence $(v_j^\nu)_\nu$ of regular values for f_j such that $\lim\limits_{\nu \to \infty} v_j^\nu = O$. Then for any ν sufficiently large, $f_j^{-1}(v_j^\nu) = \{w_{1,j}^\nu, \dots, w_{m_j,j}^\nu\} \subset \overline{B(p_j, r)} \setminus \{p_j\}$, where the points $w_{1,j}^\nu, \dots, w_{m_j,j}^\nu$ are distinct. The multiplicity at each point $w_{l,j}^\nu$ of the germ $f_j - v_j^\nu$ is equal to 1.

In the case where $m_j = 1$, we denote $v_j^\nu = O$ and $w_{1,j}^\nu = p_j$ for any ν. Then we can introduce P_ν the finite set $\{(w_{1,1}^\nu, f_1 - v_1^\nu, c_1), \dots, (w_{m_1,1}^\nu, f_1 - v_1^\nu, c_1), \dots, (w_{1,k}^\nu, f_k - v_k^\nu, c_k), \dots, (w_{m_k,k}^\nu, f_k - v_k^\nu, c_k)\}$ of poles $w_{l,j}^\nu$ in D associated respectively with the germs of the holomorphic mappings $f_j^\nu := f_j - v_j^\nu$ and the weights c_j, for $1 \leq j \leq k$ and $1 \leq l \leq m_j$.

Theorem 4.4. *If* D *is a bounded hyperconvex domain in* $\mathbf{C}^n$, *then* $g_D(P_\nu, .)$ *converges uniformly on any compact set of* $\overline{D} \setminus \{p_1, \dots, p_k\}$ *to the function* $g_D(P, .)$ *when* ν *tends to* ∞. *In addition, for any* ν, $\int_D (dd^c g_D(P, .))^n = \int_D (dd^c g_D(P_\nu, .))^n$.

Proof. We use the same notations as above. There exists $r > 0$ such that the balls $\overline{B(p_j, r)}$ are disjoint and there exist (see Proposition 4.2) two constants

α_1 and α_2 such that for any $j = 1, \dots, k$

$$c_j \log ||f_j(z)|| + \alpha_1 \le g_D(P, z) \le c_j \log ||f_j(z)|| + \alpha_2 \text{ on } \overline{B(p_j, r)}.$$

Let $\eta > 0$ be fixed. Then there exists a constant d_1 such that for any ν sufficiently large and for any $j = 1, \dots, k$, we have $f_j^{-1}(v_j^\nu) = \{w_{1,j}^\nu, \dots, w_{m_j,j}^\nu\} \subset B(p_j, r)$ and

$$(1+\eta)(c_j \log ||f_j(z)|| + \alpha_1) \ge c_j \log ||f_j^\nu(z)|| + d_1 \text{ on } \partial B(p_j, r).$$

There exists a real number $0 < r_1 \ll r$ such that for any ν sufficiently large and for any $j = 1, \dots, k$, we have

$$(1+\eta)(c_j \log ||f_j(z)|| + \alpha_2) \le c_j \log ||f_j^\nu(z)|| + d_1 \text{ on } \partial B(p_j, r_1).$$

Let v denote the psh function on D defined by $v(z) =$

$$\begin{cases} c_j \log ||f_j^\nu(z)|| + d_1 & \text{on } B(p_j, r_1), \\ & \text{for each } j \in \{1, \dots, k\}, \\ \max\{c_j \log ||f_j^\nu(z)|| + d_1, (1+\eta) g_D(P, z)\} & \text{on } B(p_j, r) \setminus B(p_j, r_1), \\ & \text{for each } j \in \{1, \dots, k\}, \\ (1+\eta) g_D(P, z) & \text{on } D \setminus \cup_j B(p_j, r). \end{cases}$$

Then $v < 0$ on D and by definition of $g_D(P_\nu, .)$, $v \le g_D(P_\nu, .)$ on D. In addition, $v \ge (1+\eta) g_D(P, .)$ on $D \setminus (\cup_j B(p_j, r_1))$; consequently $g_D(P_\nu, .) \ge (1+\eta) g_D(P, .)$ on $D \setminus (\cup_j B(p_j, r_1))$.

Conversely, we can prove that for any $\eta > 0$, there exists a real number $0 < r_2 \ll r$ such that for any ν sufficiently large, we have $g_D(P, .) \ge (1+\eta) g_D(P_\nu, .)$ on $D \setminus (\cup_j B(p_j, r_2))$.

We remark that r_1 and r_2 tend to 0 and ν tends to ∞ when η tends to 0. The proof is complete. □

Proof of Proposition 5. This is a direct consequence of Proposition 4.3. Indeed, it is well known that the pluricomplex Green function $g_{P_0}(O, .)$ in P_0 with a logarithmic pole in O and a weight equal to 1 is defined by $g_{P_0}(O, z) = \sup_{1 \le l \le n} \log(|z_l|/r_0)$, on $\overline{P_0}$. Thus, since $F : \tilde{P}(-\epsilon + \beta(\epsilon)) \to P_0$ is a proper holomorphic mapping and $F^{-1}(O)$ is the finite set of points Z, we deduce that $g_{\tilde{P}(-\epsilon+\beta(\epsilon))}(P, .) = v$, on $\overline{\tilde{P}(-\epsilon + \beta(\epsilon))}$. □

5. Proofs of Theorems A and B

5.1. The case where D is a strictly hyperconvex domain

5.1.1. K regular. Here we prove Theorem A when D is strictly hyperconvex. Let us denote by $\tilde{u}$ the following psh function on D_j:

$$\tilde{u}(z) = \sup_{1 \le l \le n} \frac{1}{p} \log |f_l(z)|.$$

Since each holomorphic function $f_l \in \mathcal{O}(D_j) \cap \mathcal{E}_{p,2j}$, we have: $\tilde{u} \leq u$ on $\overline{D}$. In addition, $\log(r_0/r_1) = p(1-2\epsilon)$. Consequently, if v is defined as in (1.7), we have $(1-2\epsilon)v = \tilde{u} + \epsilon - \beta(\epsilon)$ on D_j and

$$v - \frac{\epsilon - \beta(\epsilon)}{1-2\epsilon} \leq \frac{u}{1-2\epsilon} \text{ on } \overline{D}. \tag{5.20}$$

Let us denote by v_1 the pluricomplex Green function on D, $g_D(P, .)$, with the same logarithmic poles as $v = g_{\tilde{P}(-\epsilon+\beta(\epsilon))}(P, .)$.

Lemma 5.1.

$$\int_D (dd^c v_1)^n = \int_{\tilde{P}(-\epsilon+\beta(\epsilon))} (dd^c v)^n \to C(K, D) \textit{ when } \epsilon \to 0.$$

Proof. By definition of $g_D(P, .)$, since the function $v - \frac{\epsilon-\beta(\epsilon)}{1-2\epsilon}$ is negative on D and has logarithmic poles in P, we have $v_1 \geq v - \frac{\epsilon-\beta(\epsilon)}{1-2\epsilon}$ on $\overline{D}$. Then,

$$(1-2\epsilon)v_1 \geq -(\epsilon - \beta(\epsilon)), u \leq -\epsilon + \epsilon^2 \text{ on } \partial\tilde{P}(-\epsilon + \beta(\epsilon))$$

and

$$v_1 = u = 0 \text{ on } \partial D.$$

Since v_1 is maximal on $D \setminus \overline{\tilde{P}(-\epsilon + \beta(\epsilon))}$, we obtain that

$$v_1 \geq \frac{\epsilon - \beta(\epsilon)}{1-2\epsilon} . \frac{u}{\epsilon - \epsilon^2} = c_\epsilon u \text{ on } D \setminus \overline{\tilde{P}(-\epsilon + \beta(\epsilon))},$$

where $c_\epsilon = \dfrac{1 - \beta(\epsilon)/\epsilon}{(1-2\epsilon)(1-\epsilon)}$. As $\beta(\epsilon) \leq \epsilon^2/2$, c_ϵ converges to 1 when ϵ tends to 0. On the other hand, $v_1 \leq v$ on $\tilde{P}(-\epsilon + \beta(\epsilon))$, because $\tilde{P}(-\epsilon + \beta(\epsilon)) \subset D$. Thus $v_1 \leq -1$ on K and $v_1 \leq u$ on D. We have two continuous exhaustion functions v_1 and u on D, such that $c_\epsilon u \leq v_1 \leq u$ in a neighbourhood of ∂D. Now by applying the following Lemma 5.2, which is a special case of a more general result proved by Demailly ([Dem85], [Dem87]), we deduce that

$$C(K, D) = \int_D (dd^c u)^n \leq \int_D (dd^c v_1)^n \leq c_\epsilon^n \int_D (dd^c u)^n = c_\epsilon^n C(K, D).$$

This completes the proof of Lemma 5.1. □

Lemma 5.2. *"Comparison Theorem" – Let D be a bounded hyperconvex domain in $\mathbf{C}^n$, let u and $v \in PSH(D) \cap \mathcal{C}(\overline{D}, [-\infty, 0])$ be two exhaustion functions in D. Suppose that $u < v$ in a neighbourhood of ∂D. Then*

$$\int_D (dd^c v)^n \leq \int_D (dd^c u)^n.$$

We recall that $Z' = \{p_j : 1 \leq j \leq k'\}$ (where $k' \leq k$) is the finite set of zeros of the holomorphic mapping F in $\tilde{P}(-1+\epsilon+\beta(\epsilon))$ and that F has no zero in $\partial\tilde{P}(-1+\epsilon+\beta(\epsilon))$. Let us consider the finite set $P' = \{(p_j, F, \frac{1}{\log(r_0/r_1)}), 1 \leq j \leq k'\}$ in $\tilde{P}(-1+\epsilon+\beta(\epsilon))$ of poles p_j associated respectively with the germs of the holomorphic map F and the weights $\frac{1}{\log(r_0/r_1)}$. Let denote v_2, resp. v_3, the pluricomplex Green function $g_D(P', .)$ on D, resp. $g_{D(-1+\epsilon+\epsilon^{3/2})}(P', .)$ on $D(-1+\epsilon+\epsilon^{3/2})$.

Lemma 5.3.

$$\int_D (dd^c v_2)^n \to C(K, D) \textit{ when } \epsilon \to 0.$$

Proof. According to the previous inequality (5.20), we have

$$(1-2\epsilon)v \leq u+\epsilon-\beta(\epsilon) = -1+2\epsilon+\epsilon^{3/2}-\beta(\epsilon) \text{ on } \partial D(-1+\epsilon+\epsilon^{3/2}).$$

If we denote by d_ϵ the following positive constant: $d_\epsilon = \frac{\epsilon^{3/2}-\beta(\epsilon)}{1-2\epsilon}$, we obtain in particular that $v+1-d_\epsilon \leq 0$ on $D(-1+\epsilon+\epsilon^{3/2}) \supset \overline{\tilde{P}(-1+\epsilon+\beta(\epsilon))}$. Consequently, by the definition of v_3, $v_3 \geq v+1-d_\epsilon$ on $\overline{D}(-1+\epsilon+\epsilon^{3/2})$,

$$v_3 \geq -d_\epsilon \text{ on } \partial\tilde{P}(-1+\epsilon+\beta(\epsilon)) \text{ and } v_3 = 0 \text{ on } \partial D(-1+\epsilon+\epsilon^{3/2}).$$

Also,

$$\begin{aligned} u+1-\epsilon-\epsilon^{3/2} &\leq \epsilon^2-\epsilon^{3/2} \text{ on } \partial\tilde{P}(-1+\epsilon+\beta(\epsilon)) \\ \text{and } u+1-\epsilon-\epsilon^{3/2} &= 0 \text{ on } \partial D(-1+\epsilon+\epsilon^{3/2}). \end{aligned}$$

Since v_3 is maximal on $D(-1+\epsilon+\epsilon^{3/2}) \setminus \overline{\tilde{P}(-1+\epsilon+\beta(\epsilon))}$, we obtain that

$$v_3 \geq d_\epsilon . \frac{u+1-\epsilon-\epsilon^{3/2}}{-\epsilon^2+\epsilon^{3/2}} \text{ on } D(-1+\epsilon+\epsilon^{3/2}) \setminus \overline{\tilde{P}(-1+\epsilon+\beta(\epsilon))}.$$

We remark that $\frac{d_\epsilon}{-\epsilon^2+\epsilon^{3/2}}$ tends to 1 when ϵ tends to 0. By applying Lemma 5.2, we deduce that

$$\begin{aligned} \int_{D(-1+\epsilon+\epsilon^{3/2})} (dd^c v_3)^n &\leq \left(\frac{d_\epsilon}{-\epsilon^2+\epsilon^{3/2}}\right)^n \int_{D(-1+\epsilon+\epsilon^{3/2})} (dd^c u)^n \\ &= \left(\frac{d_\epsilon}{-\epsilon^2+\epsilon^{3/2}}\right)^n C(K, D), \end{aligned}$$

where the last equality arises from the fact that u is maximal on $D \setminus K$.

On the other hand, $v_3 \leq g_{\tilde{P}(-1+\epsilon+\beta(\epsilon))}(P', .) = v+1$ on $\tilde{P}(-1+\epsilon+\beta(\epsilon))$, because $\tilde{P}(-1+\epsilon+\beta(\epsilon)) \subset D(-1+\epsilon+\epsilon^2) \subset D(-1+\epsilon+\epsilon^{3/2})$. Thus we have on $\partial D(-1+\epsilon^2) \subset D(-1+\epsilon) \subset \tilde{P}(-1+\epsilon+\beta(\epsilon))$

$$v_3 \leq \frac{u+\epsilon-\beta(\epsilon)}{1-2\epsilon}+1 = \frac{-\epsilon+\epsilon^2-\beta(\epsilon)}{1-2\epsilon} = \frac{\epsilon(-1+\epsilon-\beta(\epsilon)/\epsilon)}{1-2\epsilon}$$

and

$$u+1-\epsilon-\epsilon^{3/2}=\epsilon^2-\epsilon-\epsilon^{3/2}=\epsilon(-1+\epsilon-\epsilon^{1/2}).$$

Consequently, according to the maximality of u on $D\setminus K$, we deduce that

$$v_3 \leq \frac{(u+1-\epsilon-\epsilon^{3/2})(1-\epsilon+\beta(\epsilon)/\epsilon)}{(1-2\epsilon)(1-\epsilon+\epsilon^{1/2})} \text{ on } D(-1+\epsilon+\epsilon^{3/2})\setminus D(-1+\epsilon^2).$$

Let us denote by e_ϵ the following positive constant: $e_\epsilon = \dfrac{(1-\epsilon+\beta(\epsilon)/\epsilon)}{(1-2\epsilon)(1-\epsilon+\epsilon^{1/2})}$. We remark that this constant e_ϵ tends to 1 when ϵ tends to 0. By applying again Lemma 5.2, we deduce that

$$\int_{D(-1+\epsilon+\epsilon^{3/2})}(dd^c v_3)^n \geq e_\epsilon^n C(K,D);$$

which completes the proof of Lemma 5.3, since $\int_D (dd^c v_2)^n = \int_{D(-1+\epsilon+\epsilon^{3/2})}(dd^c v_3)^n$. □

Lemma 5.4. *There exist $a_\epsilon > 0$ and $b_\epsilon > 0$ which converge to 0 when ϵ tends to 0 such that*

$$v_2 \geq (1+a_\epsilon)u \text{ on } \overline{D}\setminus D(-1+b_\epsilon).$$

Proof. Since $P' \subset P$, $v_2 \geq v_1$ on D. Then $(1-2\epsilon)v_2 \geq (1-2\epsilon)v-(\epsilon-\beta(\epsilon))$ on D, and in particular, $(1-2\epsilon)v_2 \geq -1+\epsilon+\beta(\epsilon)$ on $\partial\tilde{P}(-1+\epsilon+\beta(\epsilon))$. As v_2 is maximal on $D\setminus\overline{\tilde{P}(-1+\epsilon+\beta(\epsilon))}$, we deduce that $(1-2\epsilon)v_2 \geq -1+\epsilon+\beta(\epsilon)$ on $\overline{D}\setminus\overline{\tilde{P}(-1+\epsilon+\beta(\epsilon))}$, and in particular on $\partial D(-1+\epsilon+\epsilon^2)$, where $u=-1+\epsilon+\epsilon^2$. Consequently, since v_2 is maximal on $D\setminus\overline{D(-1+\epsilon+\epsilon^2)}$, we deduce that

$$v_2 \geq \frac{1-\epsilon-\beta(\epsilon)}{1-2\epsilon}.\frac{u}{1-\epsilon-\epsilon^2} \text{ on } \overline{D}\setminus D(-1+\epsilon+\epsilon^2),$$

which completes the proof of the lemma. □

Proposition 5.5. *There exists a sequence $(g_m)_m$, of classical pluricomplex Green functions on D which converges uniformly on any compact set of the form $\overline{D}\setminus D(-1+\delta)$ (where $\delta > 0$ is as small as we want) to u when m tends to infinity.*

Proof. According to Lemma 5.3, we know that $\int_D (dd^c v_2)^n = \int_{\tilde{P}(-1+\epsilon+\beta(\epsilon))}(dd^c v)^n$ tends to $C(K,D)$ when ϵ tends to 0.

In addition, according to Theorem 4.4, we know that there exists a sequence $(\tilde{g}_m)_m$ of classical pluricomplex Green functions on D with logarithmic poles in the open set $\tilde{P}(-1+\epsilon+\beta(\epsilon))$ that converges uniformly on any compact set of $\overline{D}\setminus Z'$ (we recall that the poles of v_2 are in $Z' \subset \tilde{P}(-1+\epsilon+\beta(\epsilon)) \subset D(-1+\epsilon+\epsilon^2)$) to the function v_2. Also,

$\int_D (dd^c v_2)^n = \int_D (dd^c \tilde{g}_m)^n$ for any m. Thus, according to Lemma 5.4, there exists a classical pluricomplex Green function $\tilde{g}_m$ on D (for m sufficiently large) which satisfies $\tilde{g}_m(z) \geq -1 - a'_\epsilon$, on $\overline{D} \setminus D(-1 + b_\epsilon)$, where $a'_\epsilon > 0$ converges to 0 when ϵ tends to 0.

Then, the lemma is a direct consequence of the following theorem, which is a particular case of a more general result of [NP01]. □

Theorem 5.6. *Let D be a strictly hyperconvex domain in $\mathbf{C}^n$ containing a regular compact set K. If we have a sequence of positive numbers $(\epsilon_j)_j$ which converges to 0, and a sequence $(g_j)_j$ of classical pluricomplex Green functions on D such that:*

(i) for each j, the poles of g_j are contained in $D(-1 + \epsilon_j)$,
(ii) for each j, $g_j(z) \geq -1$ on $\partial D(-1 + \epsilon_j)$,
(iii) $\int_D (dd^c g_j)^n$ converges to $C(K, D)$ when j tends to ∞,

then the sequence $(g_j)_j$ converges uniformly on any compact set of the form $\overline{D} \setminus D(-1 + \delta)$ (where $\delta > 0$ is as small as we want) to $u_{K,D}$ when j tends to ∞.

We postpone the proof of this theorem to the end of this paper. According to Proposition 5.5 and to the fact that u and g_j are maximal on $D \setminus \overline{D}(-1+\delta)$ for j sufficiently large, we deduce finally Theorem A. □

5.1.2. K strictly regular. Now we prove Theorem B. K is a strictly regular compact set in D, i.e. the closure of a relatively compact open subset ω in D such that $u_{K,D} \equiv u^*_{\omega,D}$.

Let us denote by $\omega_{-\delta} = \{z \in \omega : \mathrm{dist}(z, \partial\omega) > \delta\}$ the open set in D, defined for all sufficiently small positive constant δ, and by $K_{-\delta}$ its closure. Since $K_{-\delta}$ is a union of closed balls and D is a bounded hyperconvex domain in $\mathbf{C}^n$, we deduce that $u_{K_{-\delta},D}$ is continuous on $\overline{D}$ (see [Kli91] Corollary 4.5.9). When $\delta > 0$ decreases to 0, the family of compact sets $(K_{-\delta})_\delta$ increases to $\cup_\delta K_{-\delta} = \omega$ and the family of psh functions $(u_{K_{-\delta},D})_\delta$ decreases and converges pointwise on D to the psh function $u^*_{\omega,D}$. By hypothesis, $u := u_{K,D} \equiv u^*_{\omega,D}$. Thus, according to Dini's theorem, we obtain that this family $(u_{K_{-\delta},D})_\delta$ converges uniformly on $\overline{D}$ to u.

The open subset $\omega_{-\delta}$ is relatively compact in ω. Thus, according to a property of Narasimhan (see [Nar71], p. 116), we deduce that the holomorphically convex hull of $K_{-\delta}$ in D is also included in the interior of the holomorphically convex hull of K in D.

For any $\epsilon > 0$, there exists $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$,

$$\sqrt{1+\epsilon}.u_{K_{-\delta},D} \leq u \leq u_{K_{-\delta},D} \text{ on } \overline{D}.$$

Then, for $\delta = \delta_0$ fixed, we apply Theorem A to the couple $(K_{-\delta}, D)$, and we obtain that: for any $\epsilon' > 0$ sufficiently small, there exists a function g which is a classical pluricomplex Green function on D with a finite number of logarithmic poles, such that

(i) the poles of g are in the open neighbourhood $D(-1+\epsilon')^{\delta} = \{z \in D : u_{K_{-\delta},D}(z) < -1+\epsilon'\}$ of $(K_{-\delta})^{\wedge D}$,
(ii) g satisfies the following uniform estimates on $\overline{D} \setminus D(-1+\epsilon')^{\delta}$

$$\sqrt{1+\epsilon}.g(z) \leq u_{K_{-\delta},D}(z) \leq (1-\epsilon)g(z).$$

The family $(D(-1+\epsilon')^{\delta})_{\epsilon'}$ is a basis of neighbourhoods of $(K_{-\delta})^{\wedge D}$. Thus, for $\epsilon' > 0$ sufficiently small, $D(-1+\epsilon')^{\delta} \subset (\hat{K}_D)^{\circ}$, and the proof of Theorem B is achieved. □

5.2. The case where D is a bounded hyperconvex domain

Finally we prove Theorem A in the case of a bounded hyperconvex domain D in $\mathbf{C}^n$ containing a compact set K.

In the case where K is regular in D, $u_{K,D} = u$ is a continuous exhaustion function for D. For any $\delta > 0$ sufficiently small, $D(-\delta) = \{z \in D : u(z) < -\delta\}$ is a strictly hyperconvex domain and K is a regular compact set in $D(-\delta)$. We also have the following equality:

$$u_{K,D(-\delta)} = \max\left\{\frac{u+\delta}{1-\delta}, -1\right\} \text{ on } D(-\delta).$$

Thus, for any $\epsilon > 0$ and $\delta > 0$ sufficiently small, there exists $0 < \delta_0 < \delta$ such that for any $0 < \delta' < \delta_0$, we have

$$(1+\epsilon)^{1/3} u_{K,D(-\delta')} \leq u \leq u_{K,D(-\delta')} \text{ on } \overline{D(-\delta)}.$$

We fix $0 < \delta' \leq \inf\{\delta_0, \delta^2\}$. If we apply Theorem A to the couple $(K, D(-\delta'))$ ($u_{K,D(-\delta')} < -1+\delta'$ exactly where $u < -1+\delta'(1-\delta')$), we obtain that there exists a pluricomplex Green function g on $D(-\delta')$ which satisfies

(i) the poles of g are in the open neighbourhood $D(-1+\delta'(1-\delta'))$ of $(K)^{\wedge D}$,
(ii) g satisfies the following uniform estimates on $\overline{D(-\delta')} \setminus D(-1+\delta'(1-\delta'))$:

$$(1+\epsilon)^{1/3} g \leq u_{K,D(-\delta')} \leq (1-\epsilon)^{1/3} g.$$

If we combine these last two inequalities, we obtain that

$$(1+\epsilon)^{2/3} g \leq u \leq (1-\epsilon)^{2/3} g \text{ on } \overline{D(-\delta')} \setminus D(-1+\delta'(1-\delta')).$$

The problem now is to replace this function g by a pluricomplex Green function on D. Let us introduce G, the pluricomplex Green function on D with the same logarithmic poles as g but with weights all multiplied by the same positive constant $\frac{\delta-\delta'}{\delta}$. We remark that since δ' has been chosen $\leq \delta^2$, this constant $\frac{\delta-\delta'}{\delta}$ tends to 1, when δ tends to 0. Since $D(-\delta') \subset D$,

$G \leq \frac{\delta-\delta'}{\delta} g$ on $D(-\delta')$. Let us introduce the following function v on D by

$$v(z) = \begin{cases} \frac{\delta-\delta'}{\delta} g(z) - \frac{\delta'}{(1-\epsilon)^{2/3}} & \text{on } \overline{D(-\delta)} \\ \max\left\{\frac{\delta-\delta'}{\delta} g(z) - \frac{\delta'}{(1-\epsilon)^{2/3}}, \frac{u(z)}{(1-\epsilon)^{2/3}}\right\} & \text{on } D(-\delta') \setminus \overline{D(-\delta)} \\ \frac{u(z)}{(1-\epsilon)^{2/3}} & \text{on } D \setminus D(-\delta') \end{cases}$$

In addition,

$$u(z) = -\delta \text{ and } -\delta \leq \frac{(\delta-\delta')(1-\epsilon)^{2/3}}{\delta} g(z) - \delta' \text{ on } \partial D(-\delta)$$

and

$$u(z) = -\delta' \text{ and } \frac{(\delta-\delta')(1-\epsilon)^{2/3}}{\delta} g(z) - \delta' = -\delta' \text{ on } \partial D(-\delta').$$

Consequently, v is negative, psh on D and continuous on $\overline{D}$. According to the definition of G on D, we deduce that $v \leq G$ on $\overline{D}$, and in particular on $\overline{D(-\delta)}$, we have $G \geq \frac{\delta-\delta'}{\delta} g(z) - \frac{\delta'}{(1-\epsilon)^{2/3}}$. Thus, we have on $\overline{D(-\delta)} \setminus D(-1+\delta'(1-\delta'))$:

$$\frac{\delta(1+\epsilon)^{2/3}}{\delta-\delta'} G(z) \leq u(z) \leq \frac{\delta(1-\epsilon)^{2/3}}{\delta-\delta'} \left(G(z) + \frac{\delta'}{(1-\epsilon)^{2/3}}\right).$$

Since u and G are maximal on $D \setminus D(-\delta)$, we have the same inequalities on $\overline{D} \setminus D(-\delta)$.

As $\delta'(1-\delta') \leq \delta$, if we choose δ' sufficiently small such that $\frac{\delta}{\delta-\delta'} \leq (1+\epsilon)^{1/3}$ and $\frac{\delta\delta'}{\delta-\delta'} + \frac{\delta(1-\epsilon)^{2/3}}{\delta-\delta'} G \leq (1-\epsilon) G$ on $\overline{D} \setminus D(-1+\delta)$, we can conclude. □

In the case where K is not necessarily regular in D, for any $\delta > 0$ sufficiently small, let K^δ denote the compact subset of D defined by $K^\delta = \{z \in D : dist(z, \hat{K}_D) \leq \delta\}$.

Proof of Proposition 6. Let us denote by S the following negligible set $\{z \in D : u(z) < u^*(z)\}$ of D. By definition $u = u_{K,D} = u^*_{K,D}$ on $D \setminus S$ and u is lower semicontinuous on D. In addition, u^* is upper semicontinuous on D. Then u is continuous on $D \setminus S$. It is well known ([BT76]) that S is pluripolar and in particular, for any $\beta > 0$, there exists an open subset ω in D containing S such that $C(\omega, D) < \beta$ and u is a continuous function on the compact set $\bar{D} \setminus \omega$.

For any $\gamma > 0$ sufficiently small, K^γ is regular in D ([Kli91], Corollary 4.5.9) and the family $(u_{K^\gamma,D})_\gamma$ increases to u when γ decreases to 0. In particular, this family $(u_{K^\gamma,D})_\gamma$ converges uniformly to u on $\bar{D} \setminus \omega$.

Then by applying Theorem A to the couple (K^γ, D), we complete the proof of this proposition. □

Proof of Theorem 5.6

Theorem 5.6 is a particular case of a more general result due to Poletsky and myself [NP01]. Here we give a detailed proof of Theorem 5.6.

Fix $\delta > 0$ as small as we want, and let us denote by L_δ the compact set $\overline{D} \setminus D(-1+\delta)$. Our goal is to prove that the sequence $(g_j)_j$ converges uniformly to u on L_δ.

Let u_j denote the psh function on D, continuous on $\overline{D}$, defined by $u_j = \max\{g_j, -1\}$. According to hypothesis (i) and (ii) and to the fact that $(\epsilon_j)_j$ converges to 0, it is equivalent to prove that the sequence $(u_j)_j$ converges uniformly to u on L_δ.

According to hypothesis (i), u_j is maximal on $D \setminus \overline{D(-1+\epsilon_j)}$ for any j. In addition, u_j is equal to 0 on ∂D. Thus, by the definition of the relative extremal function $u_{\overline{D(-1+\epsilon_j)},D} = \max(\frac{u}{1-\epsilon_j}, -1)$ and by hypothesis (ii), we deduce that $u_j \geq u_{\overline{D(-1+\epsilon_j)},D} \geq u/(1-\epsilon_j)$ on $\overline{D}$. Integrating by parts provides us with the inequalities:

$$\begin{aligned}\int_D (-u)(dd^c u)^n &\geq (1-\epsilon_j)\int_D (-u_j)(dd^c u)^n\\ &= (1-\epsilon_j)\int_D (-u)dd^c u_j \wedge (dd^c u)^{n-1}\\ &\geq (1-\epsilon_j)^2\int_D (-u_j)dd^c u_j \wedge (dd^c u)^{n-1}\\ &= \ldots \geq (1-\epsilon_j)^{n+1}\int_D (-u_j)(dd^c u_j)^n.\end{aligned}$$

Thus

$$\begin{aligned}0 \leq \int_D \left(u_j - \frac{u}{1-\epsilon_j}\right)(dd^c u)^n &\leq (1-\epsilon_j)^n \int_D u_j (dd^c u_j)^n\\ &\quad - \int_D \frac{u}{1-\epsilon_j}(dd^c u)^n.\end{aligned}$$

In addition, $\int_D u_j (dd^c u_j)^n = -\int_D (dd^c g_j)^n$ converges to $-\int_D (dd^c u)^n = \int_D u(dd^c u)^n$, according to hypothesis (iii). Therefore for every $a > 0$, we have

$$\lim_{j\to\infty}\int_{\left\{u_j - \frac{u}{1-\epsilon_j} > a\right\}} (dd^c u)^n = 0.$$

Let us now fix $1 > \delta > 0$ and $\epsilon > 0$ such that $\epsilon|z|^2 - \delta < -\delta/2$ on D and denote $v_j = u_j + \epsilon|z|^2 - \delta$. Note that $(dd^c(\epsilon|z|^2 - \delta))^n = \epsilon^n c_n dV$, where the constant c_n depends only on n and dV is the volume form. Let $E_j = \{z \in D : \frac{u}{1-\epsilon_j} < v_j\}$. We remark that E_j is relatively compact in D. In addition, $u_j - \frac{u}{1-\epsilon_j} = v_j + \delta - \epsilon|z|^2 - \frac{u}{1-\epsilon_j} > \delta - \epsilon|z|^2 > \delta/2$ on E_j. By the subadditivity of the complex Monge-Ampère operator and the Comparison Principle [BT82], we have

$$\int_{E_j}(dd^c u_j)^n + \int_{E_j}(dd^c(\epsilon|z|^2-\delta))^n \le \int_{E_j}(dd^c v_j)^n \le \frac{1}{(1-\epsilon_j)^n}\int_{E_j}(dd^c u)^n.$$

$(1-\epsilon_j)^n \epsilon^n c_n m(E_j) \le \int_{\{u_j - \frac{u}{1-\epsilon_j} > \delta/2\}} (dd^c u)^n$. Thus $\lim_{j\to\infty} m(E_j) = 0$.

We remark that $u_j - \frac{u}{1-\epsilon_j} \le \delta$ on $D \setminus D(-\delta/2)$ for j sufficiently large. Since u is continuous on $\overline{D}$, there is $0 < r < \operatorname{dist}(\overline{D(-\delta)}, \partial D)$ such that $|u(z) - u(w)| < \delta$ when $z \in \overline{D(-\delta)}$, $w \in \overline{D}$ and $|z - w| < r$.

Clearly the set $G_j = \{z \in D : \frac{u}{1-\epsilon_j} < u_j - \delta\} \subset E_j$. If $m(G_j) < \delta m(B(O, r))$ for all $j \ge j_0$, then for $z_0 \in \overline{D(-\delta)}$ and $B = B(z_0, r)$, so we have

$$\begin{aligned}
u_j(z_0) &\le \frac{1}{m(B)}\int_B u_j(z)dV(z) \\
&= \frac{1}{m(B)}\left(\int_{B\setminus G_j} u_j(z)dV(z) + \int_{B\cap G_j} u_j(z)dV(z)\right) \\
&\le \frac{1}{m(B)}\int_{B\setminus G_j}\left(\frac{u(z)}{1-\epsilon_j} + \delta\right)dV(z) \\
&= \frac{1}{m(B)}\int_B\left(\frac{u(z)}{1-\epsilon_j} + \delta\right)dV(z) \\
&\quad - \frac{1}{m(B)}\int_{B\cap G_j}\left(\frac{u(z)}{1-\epsilon_j} + \delta\right)dV(z).
\end{aligned}$$

Since, $u \ge -1$ on $\overline{D}$, we have $u_j(z_0) \le \frac{u(z_0)+\delta}{1-\epsilon_j} + \delta + \delta(\frac{1}{1-\epsilon_j} - \delta) \le u(z_0) + 4\delta$ for $j \ge j_1$. Thus $u + 4\delta \ge u_j \ge \frac{u}{1-\epsilon_j} \ge u - \frac{\epsilon_j}{1-\epsilon_j} \ge u - \delta$ on $\overline{D(-\delta)}$, when $j \ge j_2$, and the proof is complete. □

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