ANALYTICITY AND INTEGRABILITY AS SYMBOLIC FORMS

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ANALYTICITY AND INTEGRABILITY AS SYMBOLIC FORMS

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Abstract

Symbolical character of modern since and its role in prediction of future experience first time has been formulated in a clear form by H. Hertz in his "Principles of Mechanics". Ernst Cassirer in his famous monograph "Philosophy of Symbolic Forms" extended Hertz's idea to Language and Mythology. Deep meaning of symbols has been analyzed by C. G. Jung in his analytical psychology and is related with Levy-Bruhl's concept of "participation mystique". In my talk I will discuss the role of analyticity in modern mathematics and physics and its symbolical meanings. It would be shown that strict quantitative prediction in science is associated with analytic continuation. The analytic nature of fundamental physical laws follows from hypothesis of fundamental symmetry of the Universe. In the global scale the analyticity leads to the concept of integrability. It would be shown that the starting ideas of the best mathematical methods were discovered in the process of solving integrable models.

1. Symbols as Window to the Eternity

Andrey Beliy in his article "Symbolism as world conception" formulated that "a symbol is window to the eternity". The basic principles of any science, methods and ways of posing questions and formulating conclusions, have to appear not as passive reflections of given being, but as created by human the intellectual symbols. The word "symbol" has its origin in συμβαλλω - cast together, joint together, unite, connect. And in symbolism we have the first attempt to show the eternal trough temporarily. Cognition in words necessarily transforms the chaos of original impressions to "Nature" with the set of laws. As mentioned by G. Noak "ability to understand symbols, appearing with forming of language, is key step leading human out of animals".

2. The Mechanical Mythology – Laplace's Determinacy

Symbolical character of mathematical-physical sciences most sharply expressed by Herman Hertz. In "Principles of the Mechanics stated in new link" (1894) [1] he formulated new ideal for developing science. Most close and significant problem of the natural science is prediction of the future (Great variety of approaches to this problem in the human history includes

prophecy of M. Nostradamus, I Ching – Chinese fortuneteller oracle book, Cabala,…)(Fig.1, *S. Michelspacher, Cabala, Augsburg, 1616*).

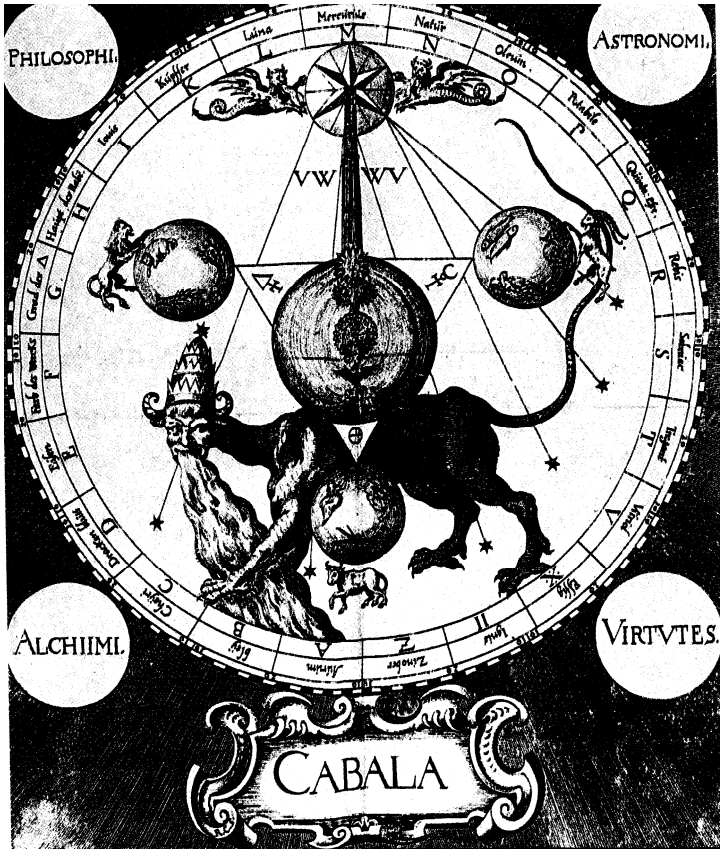

Figure 1

But in science, deduction of the future from the past is based on forming to ourselves images or symbols of external objects so that the necessary consequents of the images in thought are always the images of the necessary consequents in nature of the things pictured. In this way we are able to predict future events and check for the validity of our pictures. These images don't picture things in themselves; they picture certain structures of things, of arrangement of things. This is all we can know of reality. As mentioned by Rene Thom [2] "…it is, more or less, philosophically an illusion to distinguish between reality and metaphor".

This way we can formulate the first general meaning of analyticity, as the theoretical cognition, completely becoming symbolic construction and allowing us to **predict** events.

3. E. Cassirer's philosophy of symbolic forms in Language, Mythology and Science

E. Cassirer finds that general peculiarity inherent to language, mythos, religion, art, history and science, is **symbol, symbolical representation** [3]. The human for a long time do not live only in physical world, he lives in some kind of symbolical world. (Fig.2, *J.J. Becher, Physica subterranea,1703*). So the human transforms from **Animal rationale** to **Animal symbolicum**.

Figure 2

4. Levy-Bruhl's "participation mystique" and unconscious of a symbol

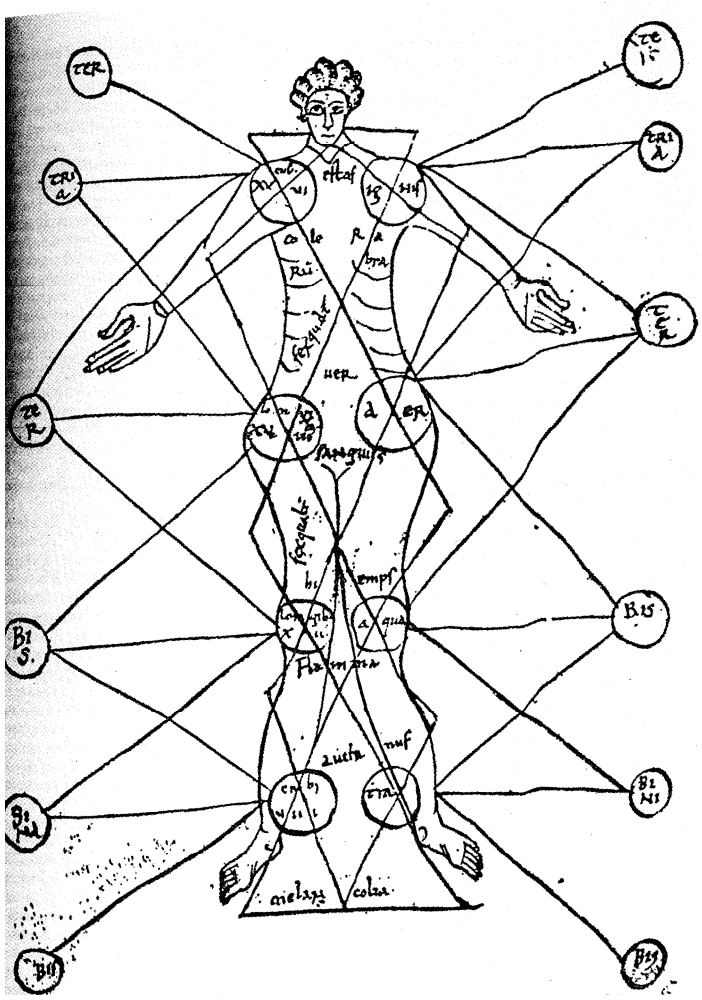

Figure 3

The ethnographies Levy-Bruhl distinguishes "participation mystique" as characteristic of primitive psychology in primitive society. But as mentioned by C. Jung, "participation mystique" is also a characteristic of any symbol [4]. Because symbols always include unconscious contents, and thus, also the human. He explains how appear symbolization of things, for example - "my watch". It is irrational, unconscious identity appearing due to the fact that all things around us are also some symbols. Symbolization occurs since: (1) every human has unconscious contents; (2) everything has its unknown sites. Now, if

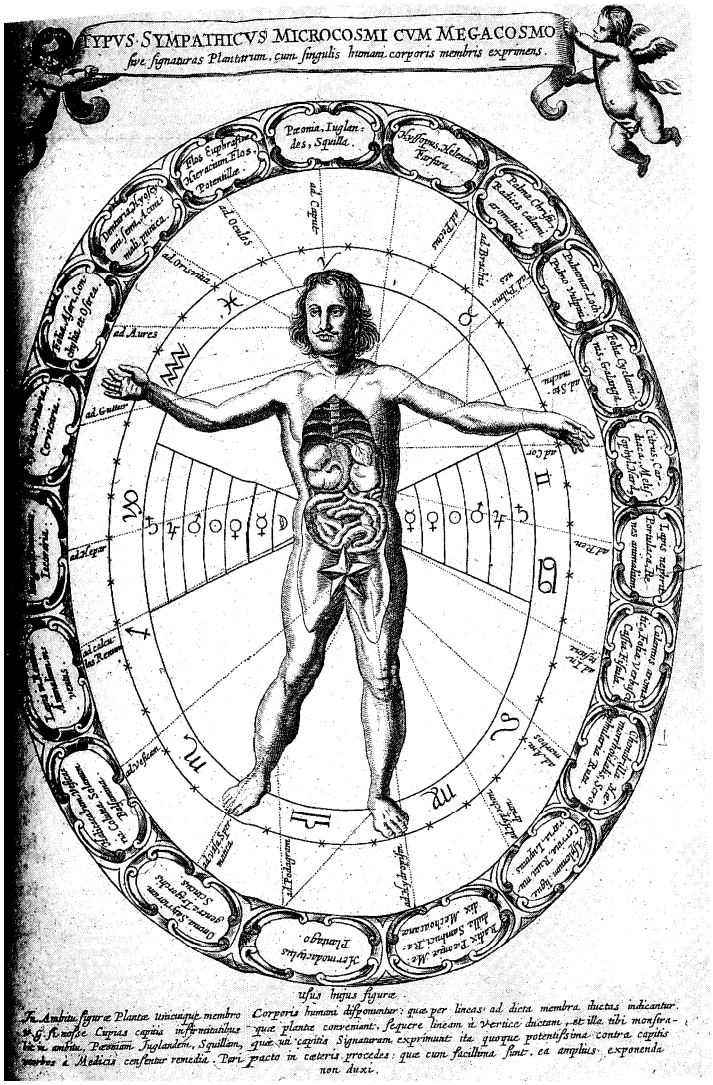

Figure 4

someone meets two unknown quantities he cannot distinguish between them. The unknown inside the human and the unknown in the thing become one. This way appears some psychic identity. In mathematics the concept of infinity has similar property (1 + 2 + 3 + 4 + … = 2 + 3 + 4 + …) or Finite **+ Infinite = Infinite**. Later Jung applied "participation mystique" concept to his Psychology of Transference (Fig3. *Burgo de Osma, 11th century*).

5. Archetyps and Mathematical Symbolism

In 1952 the famous psychoanalyst C. G. Jung and famous physicist W. Pauli published book "The Interpretation of Nature and Psyche". It contains two parts: (1) *Synchronicity* (Jung); (2) *The Influence of Archetypal Ideas on the Scientific Theories of Kepler* (Pauli) [5]. In the last article Pauli analyses transition from the earlier, magical-symbolical to the modern, quantitative-mathematical description of nature in the scientific theories of Kepler. In the Middle Ages down to the beginning of modern times, we have no natural science but just **pre-scientific stage** of a magical-symbolical description of nature. **Alchemy**, studying the **mystery** of transmutation, is a characteristic representative of this stage. The psychological significance of alchemy has been the subject of intensive investigation by C. G. Jung in his "Psychology and Alchemy" [6]. In Jung's interpretation of alchemy, the real nature of matter was not known to alchemist, he knows it only according to some feelings. Trying to use matter he projected his unconscious on unknown sites to light up them. To discover the mystery of matter, he projected another mystery – his subconscious on that has to be known ("participation mystique" and psychology of transference) (Fig.4, *A. Kircher, Mundus subterreaneus, 1682*).

Robert Fludd, respected physician and Rosicrucian from Oxford, is considered by Pauli as the representative of traditional alchemy. Fludd never distinguishes clearly between a real, material process and a symbolical representation (projection). His point of view is that true understanding of world harmony and thus also true astronomy are impossible without knowledge of the alchemical mysteries. Everything is produced without knowledge of these mysteries is an arbitrary, subjective fiction (Fig.5, *R. Fludd, Utriusque Cosmi, Vol.1, 1617*).

In contrast to this, the **intermediate stage,** represented by Johannes Kepler (1571 1630) in XVII century, is characterized by transition to the quantitative, mathematically demonstrable premises (Fig.6, *J. Kepler, Mysterium Cosmographicum, 1660*). According to Kepler, only that which is capable of quantitative, mathematical proof belongs to objective science.

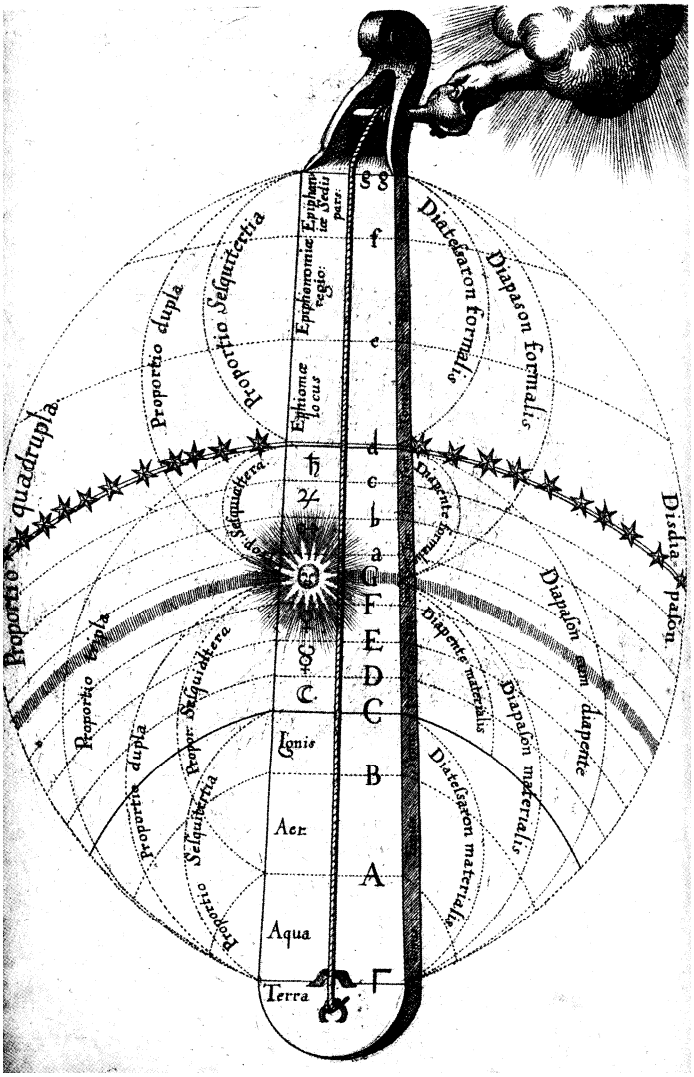

Figure 5

Kepler's views on cosmic harmony, essentially based on quantitative, mathematically demonstrable assumptions, were incompatible with the point of view of an archaic-magical description of nature represented by Fludd and result in polemic between them. At the **modern stage,** in natural sciences dominates Kepler's quantitative-mathematical description of nature. But we still are very far from completeness and unity of the world view. "Among scientists in particular the universal desire for a greater unification of our world view is greatly intensified by the fact that, though we now have natural sciences, we no longer have a total scientific picture of the world" (W. Pauli). In contrast to Kepler and Fludd, the completeness

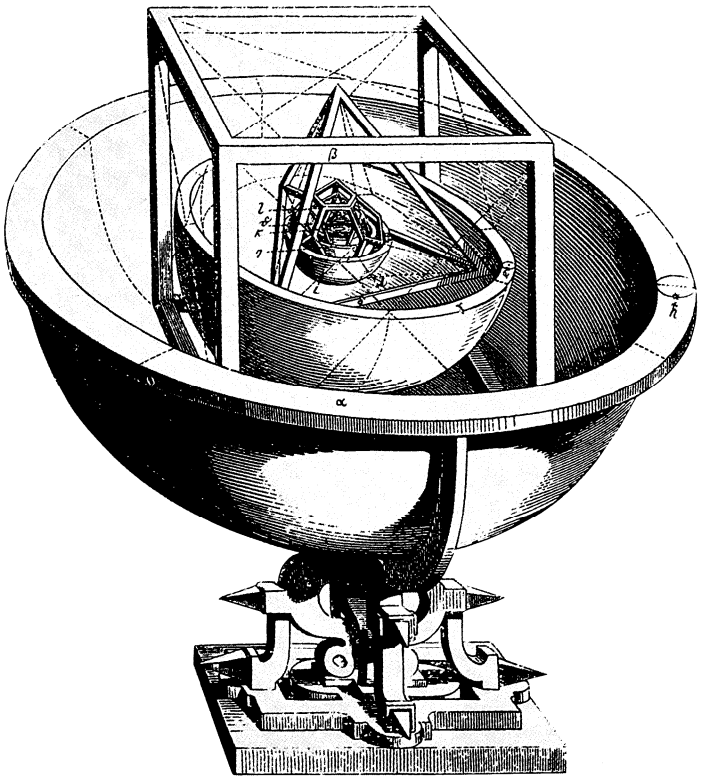

Figure 6

mean the point of view recognizing both sites of reality, quantitative and qualitative, complementarities of physical and psychical (Figs.7,8, *R. Fludd, Utriusque Cosmi, Vol.1, 1617*). It requires new concepts that are neutral with respect to opposition of Psyche and Matter. But "such concepts already exist, and they are mathematical ones: the existence of mathematical ideas, which may be applied also to the physical world (*physis*), appears possible to me because of the *homousia* (identity) of the archetypal and the physical world. In such a situation there always appears the archetype of number…it is this archetype that allows the application of mathematics to physics. On the other hand, this archetype also has a connection with psyche" (from Pauli to Jung letter).

The real number is exactly represented, pronounced and written numerical sign. Numeral, formula, notation, figure, similarly to word is a symbol – something visual and accessible to communicate. Thus according to Oswald Spengler ("Decline of the West"), the Nature is what is the subject of counting. Then, it naturally leads to the mathematical thrust of natural laws.

Figure 7

6. Symbolic Character of Natural Sciences

In the paper “Mathematical way of thinking” [7] Herman Weyl analyzes from historical point of view the symbolic character of natural sciences. In 1591 Viete introduced adequate algebraic symbols (algebra of Letters) when formulas are written without words:

$$a + b = b + a.$$

Then, Galileo discovered law of free falling body,

$$z = gt^{\wedge}2/2$$

This way he transmuted the law of Nature (characteristic of real motion), to some mathematical function constructed **a priori**. This is what modern physics is trying to do with any phenomena.

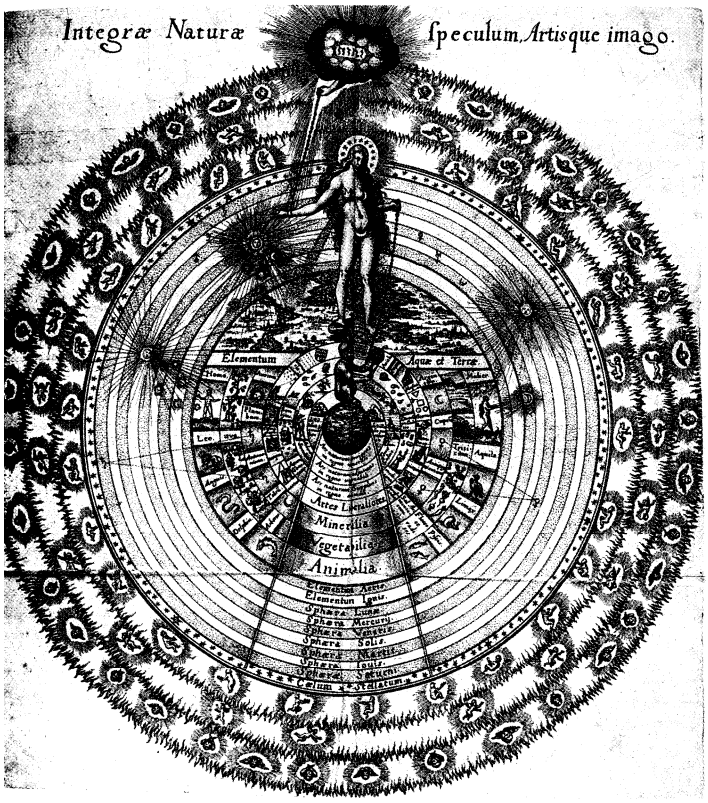

Figure 8

Leibnitz with his "Mathesis universalis" = "Characteristica generalis" = "Ars combinatorial" tried to extend algebraic symbolism to all fields of knowledge .Then,by systematic symbolism, contents of mathematics can be completely exposed without words by formulas only.

According to Spengler, the important step in development of mathematical symbolism is transition from the concept of *substance* (**ancient**), when connection of values leads to the proportion, to the concept of *function* (**modern**), when connection of relations leads to the function. Then, if the **ancient number** – is static (Pythagoras), the **western number** – is

dynamics (Descartes). Aristotle thinks in terms of substance (*actual existing characteristics*), while the concept of function extends general property rather on *all possible characteristics* (recursion n → n + 1 for integer numbers).

The existing, in this case projects on basis of the possible, precisely on variety of the possible, expanded to infinity. ("Numbers go on") [8]. Our concept of **space,** similarly to our point of view on natural numbers, is based on constructional setting of all possible places.

Analyzing the Nature we divide phenomena on simple elements visible a priori (set of space-time points), (light = monochromatic lights). So the intuitive picture has to be replaced by symbolic construction. Replacing points by their signs, mathematician changes original set to symbolic construction. Thus, characteristic – the wave length – belongs to symbolically constructed continuum of real numbers (quantitative analysis of Nature). In the Natural science we touch regions inaccessible to visual contemplation and in this case the cognition necessarily becomes symbolical. We have realized that no one of the features of our observation, even space and time, can't be preserved in world pretending on objectivity, and finally we came to the pure symbolical combinatorial construction.

Relation between given continuum and its symbolical scheme unavoidably contains the concept of *isomorphism* (participation mystique). Now, not something given in nature, like the space, but something freely created like number, becomes material for objective world construction. Important is that symbol is understandable as symbol, but not part of reality (like space or time).

Huygens has considered monochromatic light as real vibration of light ether made from special particles. We represent light by formula in which symbol F (electro-magnetic field) expressed as function of four another symbols x, y, z, t (called space-time). No one requires now that symbolic construct would considered as reality lying in the basis of phenomena. So according to Hertz: "On question, what is the Maxwell's theory, the shortest and definite answer is – the system of Maxwell equations"

Development of physics to pure symbolic construction reaches the top in XX century in Relativity Theory and Quantum Theory. The way, quantum physics represents observable quantity by Hermitian forms in infinite dimensional Hilbert space, is very characteristic example of symbolical representation.

Oskar Becker in “Das Symbolishe in der Mathematik” (1927/28) wrote: “In such a way to “interpretation” of nature one invades with completed, ontologically unclear “mathematical techniques” in hand. Like magic key, the technique opens physical problems, but only in the sense of symbolical representation, not in interpretation, really “opening” phenomena in their interrelation. Main direction of these paths is very old, archaic and even “prehistorical”:

MODERN EXACT SCIENCE BECOMES AGAIN MAGIC FROM WHICH IT WAS ONCE BORNE

6. Analyticity as Basic Principle of Mathematical Symbolism

“Complex variable theory is so beautiful that I feel that nature must have made good use of it, and, very likely, we need to make stronger use of it than we’ve done up to the present” (P. A. M. Dirac).

In his analytical psychology Jung introduces the concept of **quaternio**. Quaternio is the main organization scheme – coordinate system, instinctively applied to partitioning and ordering chaotic variety of events (the Earth directions – N, S, W, E; seasons of the year,…)(Fig.9, *Gregorius Anglus Sallwigt, Opus mago-cabalisticum, 1719*). After several letter exchanges Jung and Pauli draw following quaternio

energy

causality + synchronicity

space-time continuum

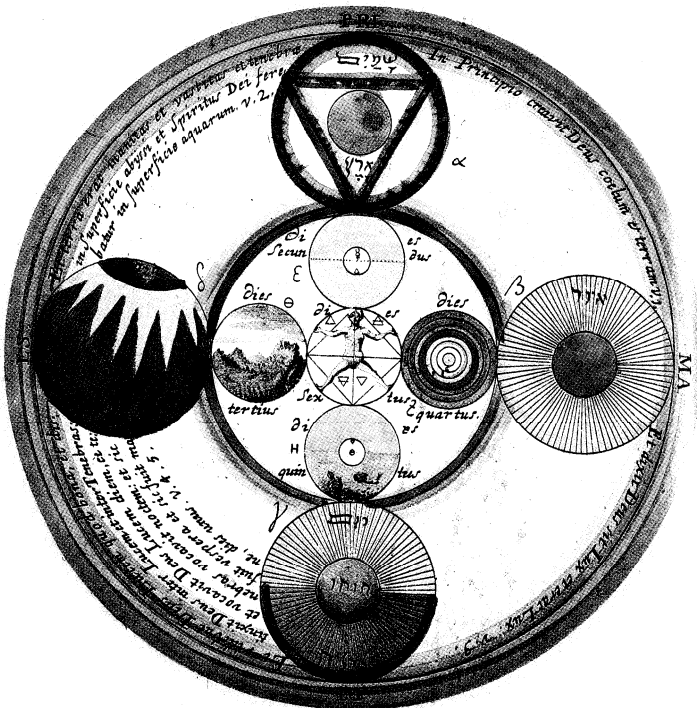

Figure 9

Analyticity property (**archetypical**) seems relate to this division – and characterizes the compatibility (possibility of free motion) in all four directions in the plane.

Some **physical** and **mathematical** meanings of analyticity:

1) Analysis as the theory of *power series*. Newton considers analysis as the studying of equations (algebraic and differential) by infinite series.

2) Causality → Analyticity → Conformal Field Theory

Our science is based on the principle of causality, and causality is considered to be an axiomatic truth. The deep relations between causality and analyticity are represented by the Dispersion Relations which are independent of particular interaction details. Dispersion relations, introduced in optics by Kronig and Kramers (1926-27) were intensively investigated in 1960s, when the concept of the S-matrix dominated in particle physics.

3) Conformal Symmetry → Infinite Dim. Group of Symmetry → Integrable System

Conformal Field Theory plays a central role in the description of phase transitions in 2-dimensional systems and in *String Theory* (the attempt at unifying all forces of Nature). It is a perfect example of systems in which the symmetries are so powerful as to allow an exact

solution of the problem (the Integrable Analytic Geometry of Quantum String). Any locally analytic function provides conformal mapping, this is why number of parameters specifying conformal transformation is infinite.

4) Amplitude of Probability in Quantum Mechanics is complex function. It has been considered as measure of distinguishability (J.A. Wheeler). Requiring the Meaning with his double requirement, of freedom to ask questions and recognition of answers, leads directly to main property of quantum theory – complex probability amplitudes. (The phase difference of these amplitudes then leads to the gauge fields).

5) Analyticity as physical principle – Atiyah-Singer Index Theorem – connection between **local** characteristics of interacting fields (Dirac's equation) with **global** characteristics of the fields (topology).

6) Lobachevskiy Geometry (Non-Euclidean Geometry) – Geometry of Analytic Functions of one complex variable (A. Poincare).

7) *Stability criterion* in vibration theory:

$$\text{Exp } (a + i\ b)\ t = \text{Exp } (at)\ \text{Cos } b\ t + i\ \text{Exp } (at)\ \text{Sin } bt$$

Complex-valued function of real variable – simple oscillation with frequency b and amplitude Exp (at). Oscillation is stable if $a < 0$ or $a = 0$. Coefficient $z = a + i\ b$ is the root of equation $f(z) = 0$, where f(z) is **analytic** (polynomial) **function**. Stability criterion – all zeroes of this polynomial located in the left half-plane.

7. Analytic continuation and quantitative prediction in Science

The problem is not only that operating with symbols is more reliable than visible thinking, but that it allows one completely legally to enter much more beyond the boundary of the region available to this thinking. Let us compare two ways analyzing the motion.

Aristotle:

throwing a stone upwards vertically; - motion up (is not natural motion – it goes against Nature), - motion down (is natural motion). **Galileo**: (great discovery!) - two motions had the same equation $Z = v\ t - g\ t^{\wedge 2}/2$. So the falling motion is the **analytic continuation** of the ascending motion. It was tremendous discovery – once we know that some phenomena are directed in their evolution by a law associated with an **analytic function,** then we are able to

make quantitative predictions. When we know **part** of a trajectory – analytic continuation gives the **whole** trajectory (canonical way of extrapolation → possibility of prediction). "I believe it is still true that, even now, strict quantitative prediction in science is associated with analytic continuation" (Rene Thom). Get out of the field of analyticity → analytic continuation is not possible → there is no strict way of extrapolation and quantitative prediction. Galilean scheme applied for computers → everything has to be finally computed → everything has to be described by essentially analytic processes. But number of phenomena with analytic descriptions is small (the measure zero). But in the fundamental physics - the laws have analytic nature. That these laws are analytic, follow from hypotheses of fundamental symmetry of the Universe. Assumption about fundamental symmetry of the Universe allows one derive laws, explaining the motion of huge phenomena (origin of Universe out of the Big Bang) – **macrocosm**, and sub-quantum level (elementary particles) – **microcosm**. Between this – the large number of phenomena do not obey the general principle of exact quantitative prediction.

8. Integrable Models in Mathematical Physics

The idea of eternal return (Pithagoreans, Demokritos, Plato, Nizshe) and cyclic time of Universe is related with analyticity (predictability) in the global sense. Mathematical models with such behavior are integrable systems. Many discoveries and developments in mathematics are connected with analytical integrable models [9].

Analytic Integrable Models in Classical Mathematics and Mechanics

- Solutions of Quadratic Algebraic Equations ($x^2 + 1 = 0$)(G. Cardano) → Complex Numbers
- Impossibility of solving Quintic or Higher Equations by root extraction (N. Abel) except in few special cases → Group Theory (E. Galois) (Which higher-degree equations admit to algebraic solution?)
- Two-body Problem (Newton) → development of the Mathematical Methods of Physics (the method of the exact analytic solution for some differential equations)

- Integrable cases of famous problems (Motion of the Top, the Heat transfer,…) → power and trigonometric series, Fourier-Laplace and other integral transforms, Complex Analysis and Symmetry arguments (XIX century)
- Integrable models with Hidden Symmetry:
  1) the motion of the top with special parameters for constant gravity (Kovalevskaya, 1889) → Riemann Surfaces and θ-functions. She introduced new approach to the problem in terms of analytic functions theory, considering time t as complex variable, find when parameters of motion are meromorphic functions of t containing 5 arbitrary constants. In addition to the Euler and the Lagrange cases she found the new one – the Kovalevskaya case. Then, Lyapunov showed that only these 3 cases exist for single-valued time dependence.
  2) the Kepler two-body problem → the Laplace-Runge-Lentz vector of integrals of motion → Hidden SO(4) symmetry group for $E < 0$ and SO(3,1) for $E > 0$. Generic central potential U(r) leads to nonperiodic orbits (Bertrand's theorem). The Kepler problem is exceptional: all orbits are closed for $E < 0$. The group of three dimensional rotations SO(3) is not enough for the periodicity of orbits. A very large hidden symmetry is needed here.

Analytic Integrable Models in Quantum and Statistical Mechanics

- Einstein's Equations in Gravity Theory : (1) the Schwarzschild solution → Black Holes; (2) the Friedman solution → Cosmological models
- Quantum Harmonic Oscillator → Lasers, the Bose-condensation and superfluidity, Quantum Field Theory
- Quantum Kepler problem → the Balmer spectrum is more degenerate (SO(4) symmetry) than the spherical symmetry requires ((SO(3))) → Mendeleev's periodic table
- One-dimensional multi-particle Quantum models (Bethe, Onsager,…) → the Bethe Ansatz, Quantum Groups, Non-commutative Geometry (Quantum Mathematics)
- The Soliton Theory .→
  1) Nonlinear waves in continuum media (plasma, nonlinear optics,…)
  2) Quantum theory: scattering theory and periodic crystals
  3) Hamiltonian dynamics, Bi-Hamiltonian structure, Symplectic Geometry

4) Algebraic Geometry of Riemann surfaces and θ-functions
5) Quantum Inverse Scattering transform
6) Yang-Baxter equations → the Jones polynomials in the theory of Knots, Topological Field Theory
7) The Hopf algebras and Quantum groups
8) Self-Dual Yang-Mills equations (instantons) → Four-dimensional topology (Donaldson, Atiyah)
9) 2-dimensional Conformal, String and Non-commutative Field Theory

Acknowledgement. This paper is devoted to my wife Svetlana, who during 25 years extends my understanding the meaning of symbols in art and life.

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